# PHY 745 Group Theory
## 11-11:50 AM MWF Olin 102

## Plan for Lecture 12:

**Continuous groups, their representations, and the great orthogonality theorem**

**Reading: Eric Carlson's lecture notes**

**Additional reference: Eugene Wigner, Group Theory**

1. Importance of commutation relations
2. Integral relations

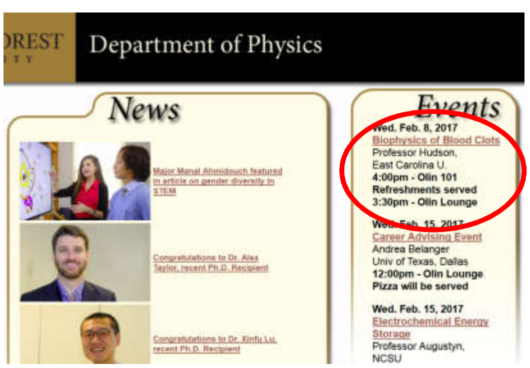


## Course schedule for Spring 2017

(Preliminary schedule -- subject to frequent adjustment.)

| | Lecture date | DDJ Reading | Topic | HW | Due date |
|---|---|---|---|---|---|
| 1 | Wed: 01/11/2017 | Chap. 1 | Definition and properties of groups | #1 | 01/20/2017 |
| 2 | Fri: 01/13/2017 | Chap. 1 | Theory of representations | | |
| | Mon: 01/16/2017 | | *MLK Holiday - no class* | | |
| 3 | Wed: 01/18/2017 | Chap. 2 | Theory of representations | | |
| 4 | Fri: 01/20/2017 | Chap. 2 | Proof of the Great Orthonality Theorem | #2 | 01/23/2017 |
| 5 | Mon: 01/23/2017 | Chap. 3 | Notion of character of a representation | #3 | 01/25/2017 |
| 6 | Wed: 01/25/2017 | Chap. 3 | Examples of point groups | #4 | 01/27/2017 |
| 7 | Fri: 01/27/2017 | Chap. 4 & 8 | Symmetry of vibrational modes | #5 | 01/30/2017 |
| 8 | Mon: 01/30/2017 | Chap. 4 & 8 | Symmetry of vibrational modes | #6 | 02/01/2017 |
| 9 | Wed: 02/01/2017 | Chap. 8 | Vibrational excitations | #7 | 02/03/2017 |
| 10 | Fri: 02/03/2017 | Notes | Continuous groups | #8 | 02/06/2017 |
| 11 | Mon: 02/06/2017 | Notes | Group of three-dimensional rotations | #9 | 02/08/2017 |
| 12 | Wed: 02/08/2017 | Notes | Continuous groups | #10 | 02/10/2017 |
| 13 | Fri: 02/10/2017 | | | | |
| 14 | Mon: 02/13/2017 | | | | |
| 15 | Wed: 02/15/2017 | | | | |





Group structure of SO(3) and SU(2)

$$O_R(\alpha,\ \hat{\mathbf{n}}) = e^{-i\alpha\mathbf{J}\cdot\hat{\mathbf{n}}/\hbar}$$

Multiplication rule:

$$O_R(\alpha_1,\ \hat{\mathbf{n}}_1)O_R(\alpha_2,\ \hat{\mathbf{n}}_2) = O_R(\alpha_3,\ \hat{\mathbf{n}}_3)$$

Example for $\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2 = \hat{\mathbf{z}}$

$$O_R(\alpha_1,\ \hat{\mathbf{z}})O_R(\alpha_2,\ \hat{\mathbf{z}}) = O_R(\alpha_3,\ \hat{\mathbf{z}})$$

$$e^{-i\alpha_1 J_z/\hbar}e^{-i\alpha_2 J_z/\hbar} = e^{-i(\alpha_1+\alpha_2)J_z/\hbar} \qquad \Rightarrow \alpha_3 = \alpha_1 + \alpha_2$$



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Example for $\hat{\mathbf{n}}_1 = \hat{\mathbf{x}} \qquad \hat{\mathbf{n}}_2 = \hat{\mathbf{y}}$

$$O_R(\alpha_1,\ \hat{\mathbf{x}})O_R(\alpha_2,\ \hat{\mathbf{y}}) = O_R(\alpha_3,\ \hat{\mathbf{n}}_3)$$

$$e^{-i\alpha_1 J_x/\hbar}e^{-i\alpha_2 J_y/\hbar} = e^{-i\left(\alpha_1 J_x + \alpha_2 J_y - i\left[\alpha_1 J_x, \alpha_2 J_y\right]/(2\hbar)\right)/\hbar}$$

In this case: $\left[J_x, J_y\right] = i\hbar J_z$



$$e^{-i\alpha_1 J_x/\hbar}e^{-i\alpha_2 J_y/\hbar} = e^{-i\left(\alpha_1 J_x + \alpha_2 J_y - i\left[\alpha_1 J_x, \alpha_2 J_y\right]/(2\hbar)\right)/\hbar}$$

$$\text{LHS}=\left(1 - i\alpha_1 J_x/\hbar + \frac{1}{2!}\left(-i\alpha_1 J_x/\hbar\right)^2 + \frac{1}{3!}\left(-i\alpha_1 J_x/\hbar\right)^3 \ldots\right)$$

$$\times\left(1 - i\alpha_2 J_y/\hbar + \frac{1}{2!}\left(-i\alpha_2 J_y/\hbar\right)^2 + \frac{1}{3!}\left(-i\alpha_2 J_y/\hbar\right)^3 \ldots\right)$$

$$=1 - i\left(\alpha_1 J_x + \alpha_2 J_y\right)/\hbar - \frac{1}{2!}\left(\left(\alpha_1 J_x\right)^2 + \left(\alpha_2 J_y\right)^2\right)/\hbar^2 - \alpha_1\alpha_2 J_x J_y/\hbar^2 + \ldots$$

$$\text{RHS}=1 - i\alpha_1 J_x/\hbar - i\alpha_2 J_y/\hbar - \left[\alpha_1 J_x, \alpha_2 J_y\right]/\left(2\hbar^2\right)$$

$$+\frac{1}{2!}\left(-i\alpha_1 J_x/\hbar - i\alpha_2 J_y/\hbar - \left[\alpha_1 J_x, \alpha_2 J_y\right]/\left(2\hbar^2\right)\right)^2 + \ldots..$$

$$=1 - i\left(\alpha_1 J_x + \alpha_2 J_y\right)/\hbar - \frac{\alpha_1\alpha_2}{2}\left(J_x J_y - J_y J_x\right)/\hbar^2 + \frac{1}{2!}\left(\left(\alpha_1 J_x\right)^2 + \left(\alpha_2 J_y\right)^2\right)/\hbar^2$$

$$-\frac{\alpha_1\alpha_2}{2}\left(J_x J_y + J_y J_x\right)/\hbar^2 + \ldots$$

$$= 1 - i\left(\alpha_1 J_x + \alpha_2 J_y\right)/\hbar - \alpha_1\alpha_2 J_x J_y/\hbar^2 + \frac{1}{2!}\left(\left(\alpha_1 J_x\right)^2 + \left(\alpha_2 J_y\right)^2\right)/\hbar^2 \ + \ldots.$$

Summary of result for $\hat{\mathbf{n}}_1 = \hat{\mathbf{x}} \quad \hat{\mathbf{n}}_2 = \hat{\mathbf{y}}$

$$O_R(\alpha_1,\ \hat{\mathbf{x}})O_R(\alpha_2,\ \hat{\mathbf{y}}) = O_R(\alpha_3,\ \hat{\mathbf{n}}_3)$$

$$e^{-i\alpha_1 J_x/\hbar}e^{-i\alpha_2 J_y/\hbar} = e^{-i\left(\alpha_1 J_x + \alpha_2 J_y - i\left[\alpha_1 J_x, \alpha_2 J_y\right]/(2\hbar)\right)/\hbar}$$

Since $\left[J_x, J_y\right] = i\hbar J_z$ :

$$e^{-i\alpha_1 J_x/\hbar}e^{-i\alpha_2 J_y/\hbar} = e^{-i\left(\alpha_1 J_x + \alpha_2 J_y + \alpha_1\alpha_2 J_z/2\right)/\hbar}$$



Now consider the great orthogonality theorem:

$$\sum_R \left(D^{\Gamma_n}(R)_{\mu\nu}\right)^* D^{\Gamma_{n'}}(R)_{\alpha\beta} = \frac{h}{l_n}\delta_{nn'}\delta_{\mu\alpha}\delta_{\nu\beta}$$

For continuous groups, the summation becomes an integral:

$$\int \left(D^{\Gamma_n}(R)_{\mu\nu}\right)^* D^{\Gamma_{n'}}(R)_{\alpha\beta}\, dR = \frac{\delta_{nn'}\delta_{\mu\alpha}\delta_{\nu\beta}}{l_n}\int dR$$



In terms of the characters of the representations:

$$\int \left(\chi^{\Gamma_n}(R)\right)^* \chi^{\Gamma_{n'}}(R)\, dR = \delta_{nn'}\int dR$$

Procedure for carrying out integration over group elements

In general, there will be continuous parameter(s) which characterize each group element $R = R(\alpha, \beta, ...)$

$\int dR \Rightarrow \int g(R(\alpha,\beta...))\, d\alpha\, d\beta...$ where $g(R(\alpha,\beta...))$ represents the density of group elements in the neighborhood of $R$ in the parameter space of $\alpha, \beta...$

Analysis of continuum properties of continuous groups, following notes of Professor Eric Carlson.

Notation:

Group elements $R, S$

Continuous parameters $x_1, x_2 ... \Rightarrow R(x_1, x_2 ...) \equiv R(\mathbf{x})$

Identity $E = R(\mathbf{e})$

Continuous group based on nearby mapping of continuous parameters



Some details from Professor Carlson's Notes:

It will often be awkward to perform multiplication using the abstract group elements. We define the coordinate product function $\boldsymbol{\mu}(\mathbf{x},\mathbf{y})$ by

$$\boldsymbol{\mu}(\mathbf{x},\mathbf{y}) \equiv R^{-1}\left(R(\mathbf{x}) \cdot R(\mathbf{y})\right), \tag{1.5}$$

where $R^{-1}$ is the inverse of $R$, which turns group elements into coordinates. In other words, multiply the group element corresponding to $\mathbf{x}$ and the group element corresponding to $\mathbf{y}$ and figure out the coordinates of the resulting group element. As an example, suppose we are working with the complex numbers, and we denote them by a pair of coordinates $R(x_1, x_2) = x_1 + ix_2$. Then we would have

$$\begin{aligned}\boldsymbol{\mu}(\mathbf{x},\mathbf{y}) &= \boldsymbol{\mu}(x_1, x_2; y_1, y_2) = R^{-1}\left[R(x_1,x_2)\cdot R(y_1,y_2)\right] = R^{-1}\left[(x_1+ix_2)(y_1+iy_2)\right] \\ &= R^{-1}\left[(x_1y_1 - x_2y_2) + i(x_1y_2 + x_2y_1)\right] = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1)\end{aligned} \tag{1.6}$$

Another example:

$R(\alpha_1) = O_R(\alpha_1, \hat{\mathbf{z}})$

$\mu(\alpha_1, \alpha_2) = R^{-1}\left(R(\alpha_1)R(\alpha_2)\right) = \alpha_1 + \alpha_2$



From the associative and identity properties of the group, it is easily proven that

$$\boldsymbol{\mu}(\mathbf{x},\mathbf{e}) = \boldsymbol{\mu}(\mathbf{e},\mathbf{x}) = \mathbf{x}, \tag{1.7a}$$

$$\boldsymbol{\mu}(\mathbf{x},\boldsymbol{\mu}(\mathbf{y},\mathbf{z})) = \boldsymbol{\mu}(\boldsymbol{\mu}(\mathbf{x},\mathbf{y}),\mathbf{z}) \tag{1.7b}$$

Use of measure function to perform needed integrals

$$\int dR\ f(R) \Rightarrow \begin{cases} \int d_R R\, f(R) = C\int d^N\mathbf{x} \left|\partial\mu_i(\mathbf{y},\mathbf{x})/\partial y_j\right|_{\mathbf{y}=\mathbf{e}}^{-1} f(R(\mathbf{x})). & (1.11a) \\ \int d_L R\, f(R) = C\int d^N\mathbf{x} \left|\partial\mu_i(\mathbf{x},\mathbf{y})/\partial y_j\right|_{\mathbf{y}=\mathbf{e}}^{-1} f(R(\mathbf{x})). & (1.11b)\end{cases}$$

Normalization constant

Jacobian for left or right measures

We are particularly interested in showing that the great orthogonality theorem holds for continuous groups which requires :

$$\int f(R)\,dR = \int f(RS)\,dR = \int f(SR)\,dR\,, \quad (1.9)$$

Proof from Professor Carlson's notes:

Let $S$ be any element of the group, then $S = R(\mathbf{s})$ for some coordinate $\mathbf{s}$. We wish to simplify the expression

$$\begin{aligned}\int d_L R\, f(S\cdot R) &= C\int d^N\mathbf{x}\left|\partial\mu_i(\mathbf{x},\mathbf{y})/\partial y_j\right|_{\mathbf{y}=\mathbf{e}}^{-1} f\left(R(\mathbf{s})\cdot R(\mathbf{x})\right)\\ &= C\int d^N\mathbf{x}\left|\partial\mu_i(\mathbf{x},\mathbf{y})/\partial y_j\right|_{\mathbf{y}=\mathbf{e}}^{-1} f\left(R(\boldsymbol{\mu}(\mathbf{s},\mathbf{x}))\right)\end{aligned} \quad (1.12)$$



Changing variables --

Now, change variables on the right side. Let $\mathbf{z} = \boldsymbol{\mu}(\mathbf{s},\mathbf{x})$, then as $\mathbf{x}$ varies over all possible coordinates of the group, so will $\mathbf{z}$, so we can replace the integral over all $\mathbf{x}$ with the integral over all $\mathbf{z}$ However, when performing such a change of variables, we know from multi-variable calculus that a Jacobian must be included in the integral, so that

$$\int d^N\mathbf{z} = \int d^N\mathbf{x}\left|\partial z_i/\partial x_j\right| = \int d^N\mathbf{x}\left|\partial\mu_i(\mathbf{s},\mathbf{x})/\partial x_j\right| \quad (1.13)$$

This allows us to rewrite (1.12) as

$$\begin{aligned}\int d_L R\, f(S\cdot R) &= C\int d^N\mathbf{z}\left|\frac{\partial\mu_i(\mathbf{s},\mathbf{x})}{\partial x_j}\right|^{-1}\left|\frac{\partial\mu_i(\mathbf{x},\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{e}}^{-1} f\left(R(\mathbf{z})\right)\\ &= C\int d^N\mathbf{z}\left|\sum_k\frac{\partial\mu_i(\mathbf{s},\mathbf{x})}{\partial x_k}\frac{\partial\mu_k(\mathbf{x},\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{e}}^{-1} f\left(R(\mathbf{z})\right)\end{aligned} \quad (1.14)$$



Using **z** varable:

$$\int d^N\mathbf{z} = \int d^N\mathbf{x}\left|\partial z_i/\partial x_j\right| = \int d^N\mathbf{x}\left|\partial\mu_i(\mathbf{s},\mathbf{x})/\partial x_j\right| \quad (1.13)$$

This allows us to rewrite (1.12) as

$$\begin{aligned}\int d_L R\, f(S\cdot R) &= C\int d^N\mathbf{z}\left|\frac{\partial\mu_i(\mathbf{s},\mathbf{x})}{\partial x_j}\right|^{-1}\left|\frac{\partial\mu_i(\mathbf{x},\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{e}}^{-1} f\left(R(\mathbf{z})\right)\\ &= C\int d^N\mathbf{z}\left|\sum_k\frac{\partial\mu_i(\mathbf{s},\mathbf{x})}{\partial x_k}\frac{\partial\mu_k(\mathbf{x},\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{e}}^{-1} f\left(R(\mathbf{z})\right)\end{aligned} \quad (1.14)$$

where we have used the fact that the product of a determinant of two matrices equals the determinant of the product. Now, consider the expression

$$\begin{aligned}\left.\frac{\partial\mu_i(\mathbf{z},\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{e}} &= \left.\frac{\partial\mu_i(\boldsymbol{\mu}(\mathbf{s},\mathbf{x}),\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{e}} = \left.\frac{\partial\mu_i(\mathbf{s},\boldsymbol{\mu}(\mathbf{x},\mathbf{y}))}{\partial y_j}\right|_{\mathbf{y}=\mathbf{e}}\\ &= \sum_k\left.\frac{\partial\mu_i(\mathbf{s},\mathbf{v})}{\partial v_k}\right|_{\mathbf{v}=\boldsymbol{\mu}(\mathbf{x},\mathbf{e})}\left.\frac{\partial\mu_k(\mathbf{x},\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{e}} = \sum_k\left.\frac{\partial\mu_i(\mathbf{s},\mathbf{x})}{\partial x_k}\frac{\partial\mu_k(\mathbf{x},\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{e}}\end{aligned} \quad (1.16)$$

When the dust clears:

$$\int d_L R\, f(S\cdot R) = C\int d^N \mathbf{z} \left|\frac{\partial \mu_i(\mathbf{z},\mathbf{y})}{\partial y_j}\right|^{-1}_{\mathbf{y}=\mathbf{e}} f(R(\mathbf{z})) = \int d_L R\, f(R) \qquad (1.17)$$

A nearly identical proof then shows that

$$\int d_R R\, f(R\cdot S) = \int d_R R\, f(R) \qquad (1.18)$$

These arguments form the basis of the extension of the great orthogonality theorem to continuous groups.
See Professor Carlson's notes for more details.