PHY 745 Group Theory
11-11:50 AM MWF Olin 102

Plan for Lecture 11:

Introduction to groups having infinite dimension

Reading: Eric Carlson's lecture notes

Additional reference: Symmetry Principles.. Melvin Lax, Wiley (1964)

1. 3-dimensional rotation group
2. Generators of the group
3. Algebraic relationships

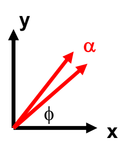
Course schedule for Spring 2017

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	DDJ Reading	Topic	HW	Due date
1	Wed: 01/11/2017	Chap. 1	Definition and properties of groups	#1	01/20/2017
2	Fri: 01/13/2017	Chap. 1	Theory of representations		
	Mon: 01/16/2017		*MLK Holiday - no class*		
3	Wed: 01/18/2017	Chap. 2	Theory of representations		
4	Fri: 01/20/2017	Chap. 2	Proof of the Great Orthonality Theorem	#2	01/23/2017
5	Mon: 01/23/2017	Chap. 3	Notion of character of a representation	#3	01/25/2017
6	Wed: 01/25/2017	Chap. 3	Examples of point groups	#4	01/27/2017
7	Fri: 01/27/2017	Chap. 4 & 8	Symmetry of vibrational modes	#5	01/30/2017
8	Mon: 01/30/2017	Chap. 4 & 8	Symmetry of vibrational modes	#6	02/01/2017
9	Wed: 02/01/2017	Chap. 8	Vibrational excitations	#7	02/03/2017
10	Fri: 02/03/2017	Notes	Continuous groups	#8	02/06/2017
11	Mon: 02/06/2017	Notes	Group of three-dimensional rotations	#9	02/08/2017
12	Wed: 02/08/2017				
13	Fri: 02/10/2017				
14	Mon: 02/13/2017				
15	Wed: 02/15/2017				

Notion of a generator for the three-dimensional rotation group

Consider rotation by an angle α about the z-axis



Let $\psi(\phi)$ denote the probability distribution for a quantum system in terms of its angle ϕ.

$$\langle\phi\rangle = \int d\phi\, \psi(\phi)^* \phi\, \psi(\phi)$$

$$\langle\phi\rangle + \alpha = \int d\phi\, \psi(\phi)^* (\phi + \alpha)\, \psi(\phi)$$

$$= \int d\phi' \psi(\phi' - \alpha)^* \phi' \psi(\phi' - \alpha)$$

Taylor expansion:

$$\psi(\phi'-\alpha)=\psi(\phi')-\alpha\frac{\partial\psi(\phi')}{\partial\phi'}+\frac{1}{2}\alpha^2\frac{\partial^2\psi(\phi')}{\partial\phi'^2}+....$$

$$=e^{-\alpha\frac{\partial}{\partial\phi'}}\psi(\phi')$$

$$=O_{-\alpha}\psi(\phi')$$

Generator operator for rotation: $O_{-\alpha}=e^{-\alpha\frac{\partial}{\partial\phi'}}$

Connection to angular momentum in quantum theory

Generator operator for rotation: $O_{-\alpha}=e^{-\alpha\frac{\partial}{\partial\phi}}$

In terms of cartesian coordinates:

$$-\frac{\partial}{\partial\phi}=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$$

Recall the classical expression for angular momentum:

$\mathbf{L}=\mathbf{r}\times\mathbf{p}$

Classical expression	Quantum operator in coordinate representation
$L_z=xp_y-yp_x$	$L_z=-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$
$L_x=yp_z-zp_y$	$L_x=-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)$
$L_y=zp_x-xp_z$	$L_y=-i\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)$

Generator operator for rotations -- continued

$$O_{-\alpha}=e^{-\alpha\frac{\partial}{\partial\phi}}=e^{-i\alpha L_z/\hbar}$$

More "standard" notation --
operator for counterclockwise
rotation about the $\hat{\mathbf{n}}$ axis by angle α :

$$O_R(\alpha,\ \hat{\mathbf{n}})=e^{-i\alpha\mathbf{L}\cdot\hat{\mathbf{n}}/\hbar}$$

Commuter relations for rotation generator operators

$$\left[L_x, L_y\right] = i\hbar L_z \qquad \left[L_y, L_z\right] = i\hbar L_x \qquad \left[L_z, L_x\right] = i\hbar L_y$$

Additional relations:

$$\left[L^2, L_i\right] = 0 \quad \text{for } i = x, y, z$$

It is convenient to find eigenfunctions of L^2 and L_z :

$$L^2 \left|lm\right\rangle = \hbar^2 l(l+1)\left|lm\right\rangle$$

$$L_z \left|lm\right\rangle = \hbar m \left|lm\right\rangle$$

$$L_\pm \left|lm\right\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}\left|l(m\pm 1)\right\rangle$$

Eigenfunctions of rotation operator

$$O_R(\alpha,\ \hat{\mathbf{n}}) = e^{-i\alpha \mathbf{L}\cdot\hat{\mathbf{n}}/\hbar}$$

$$O_R(\alpha,\ \hat{\mathbf{z}})\left|lm\right\rangle = e^{-i\alpha L_z/\hbar}\left|lm\right\rangle = e^{-i\alpha m}\left|lm\right\rangle$$

Irreducible representations in terms of angular momentum eigenfunctions

$$\chi^l(\alpha) = \sum_{m=-l}^{l} \left\langle lm\right| O_R(\alpha,\ \hat{\mathbf{z}}) \left|lm\right\rangle = \sum_{m=-l}^{l} e^{-i\alpha m} = \frac{\sin\left[\alpha\left(l+\frac{1}{2}\right)\right]}{\sin(\alpha/2)}$$

$$\text{Note that:} \quad \chi^l(\alpha + 2\pi) = (-1)^{2l}\chi^l(\alpha)$$

The group of 3-dimensional rotations about a point is called SO(3).

Group of all unitary matrices of dimension 2 – SU(2)

$$M = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{where } |a|^2 + |b|^2 = 1$$

$$\Rightarrow M = M(\alpha, n) = e^{-i\frac{1}{2}\alpha\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\left(\tfrac{\alpha}{2}\right) - i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\sin\left(\tfrac{\alpha}{2}\right)$$

$$\text{where} \quad \boldsymbol{\sigma} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x}\hat{\mathbf{x}} + \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y}\hat{\mathbf{y}} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}\hat{\mathbf{z}}$$

Group of all unitary matrices of dimension 2 – SU(2) --continued

Note that:

$$e^{-i\frac{1}{2}\alpha\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}} = 1 + \left(-i\tfrac{1}{2}\alpha\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\right) + \frac{1}{2!}\left(-i\tfrac{1}{2}\alpha\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\right)^2 + \frac{1}{3!}\left(-i\tfrac{1}{2}\alpha\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\right)^3 + \frac{1}{4!}\left(-i\tfrac{1}{2}\alpha\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\right)^4 + \ldots$$

since $(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$e^{-i\frac{1}{2}\alpha\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(1 - \frac{1}{2!}\left(\frac{\alpha}{2}\right)^2 + \frac{1}{4!}\left(\frac{\alpha}{2}\right)^4 - \ldots\right) - i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\left(\left(\frac{\alpha}{2}\right) - \frac{1}{3!}\left(\frac{\alpha}{2}\right)^3 + \ldots\right)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\left(\tfrac{\alpha}{2}\right) - i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\sin\left(\tfrac{\alpha}{2}\right)$$

Generalization of angular momentum to include ½ integers

$$[J_x, J_y] = i\hbar J_z \qquad [J_y, J_z] = i\hbar J_x \qquad [J_z, J_x] = i\hbar J_y$$

Additional relations:

$$[J^2, J_i] = 0 \quad \text{for } i = x, y, z$$

It is convenient to find eigenfunctions of J^2 and J_z :

$$J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle$$

$$J_z|jm\rangle = \hbar m|jm\rangle$$

$$J_\pm|jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j(m\pm 1)\rangle$$

Eigenfunctions of rotation-spin operator

$$O_R(\alpha, \hat{\mathbf{n}}) = e^{-i\alpha\mathbf{J}\cdot\hat{\mathbf{n}}/\hbar}$$

$$O_R(\alpha, \hat{\mathbf{z}})|jm\rangle = e^{-i\alpha J_z/\hbar}|jm\rangle = e^{-i\alpha m}|jm\rangle$$

Irreducible representations in terms of angular momentum-spin eigenfunctions

$$\chi^j(\alpha) = \sum_{m=-j}^{j}\langle jm|O_R(\alpha, \hat{\mathbf{z}})|jm\rangle = \sum_{m=-j}^{j} e^{-i\alpha m} = \frac{\sin\left[\alpha\left(j+\frac{1}{2}\right)\right]}{\sin(\alpha/2)}$$

Note that: $\chi^j(\alpha + 2\pi) = (-1)^{2j}\chi^j(\alpha)$