PHY 745 Group Theory
11-11:50 AM MWF Olin 102

Plan for Lecture 10:

Introduction to groups having infinite dimension

Reading: Eric Carlson's lecture notes

1. **Example – 3-dimensional rotation group**
2. **Some properties of continuous groups**

Course schedule for Spring 2017

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	DDJ Reading	Topic	HW	Due date
1	Wed: 01/11/2017	Chap. 1	Definition and properties of groups	#1	01/20/2017
2	Fri: 01/13/2017	Chap. 1	Theory of representations		
	Mon: 01/16/2017		*MLK Holiday - no class*		
3	Wed: 01/18/2017	Chap. 2	Theory of representations		
4	Fri: 01/20/2017	Chap. 2	Proof of the Great Orthonality Theorem	#2	01/23/2017
5	Mon: 01/23/2017	Chap. 3	Notion of character of a representation	#3	01/25/2017
6	Wed: 01/25/2017	Chap. 3	Examples of point groups	#4	01/27/2017
7	Fri: 01/27/2017	Chap. 4 & 8	Symmetry of vibrational modes	#5	01/30/2017
8	Mon: 01/30/2017	Chap. 4 & 8	Symmetry of vibrational modes	#6	02/01/2017
9	Wed: 02/01/2017	Chap. 8	Vibrational excitations	#7	02/03/2017
10	Fri: 02/03/2017	Notes	Continuous groups	#8	02/06/2017
11	Mon: 02/06/2017				
12	Wed: 02/08/2017				
13	Fri: 02/10/2017				
14	Mon: 02/13/2017				
15	Wed: 02/15/2017				

Consider a three-dimensional rotation. For example a rotation by angle α about the z-axis, transforms as follows:

$$R_\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

These rotations follow the multiplication rule

$R_\beta R_\alpha = R_\gamma$ where $\gamma = \alpha + \beta$

$$R_\beta R_\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

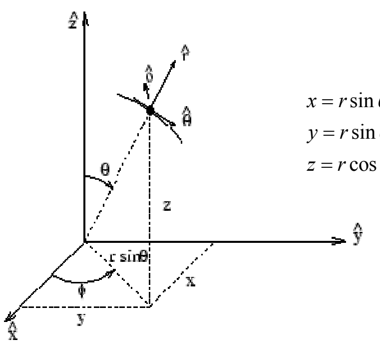
$$= \begin{pmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & 0 \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Evidently, the set of all rotations α about the z-axis form a group

1. $R_\beta R_\alpha = R_\gamma$
2. The identity exists: $E = R_{\alpha=0}$
3. Inverses exists: $R_\alpha^{-1} = R_{-\alpha}$
4. The associative relation holds: $(R_\gamma R_\beta) R_\alpha = R_\gamma (R_\beta R_\alpha)$

Thanks to Euler, we can generalize the notion to say that all three-dimensional rotations form a group

Spherical polar coordinates



$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html

Spherical harmonic basis functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\; e^{\pm i\varphi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2(\pm2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\; e^{\pm 2i\varphi}$$

$$Y_{2(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\;\cos\theta\; e^{\pm i\varphi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

It can be shown that

$$R_\alpha Y_{lm}(\theta,\phi) = \sum_{m'=-l}^{l} M^l_{mm'} Y_{lm'}(\theta,\phi)$$

Apparently: $M^0_{00} = 1$

Consider the case for $l = 1$:

$$Y_{1(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi} = \mp\sqrt{\frac{3}{8\pi}}\frac{x \pm iy}{r}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$

$$R_\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha\ x + \sin\alpha\ y \\ -\sin\alpha\ x + \cos\alpha\ y \\ z \end{pmatrix}$$

Case for l=1 – continued --

$$R_\alpha Y_{1(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{3}{8\pi}} R_\alpha\left(\frac{x \pm iy}{r}\right)$$

$$= \mp\sqrt{\frac{3}{8\pi}}\frac{\left(\cos\alpha\ x + \sin\alpha\ y\right) \pm i\left(-\sin\alpha\ x + \cos\alpha\ y\right)}{r}$$

$$= \mp\sqrt{\frac{3}{8\pi}}\left[\left(\cos\alpha \mp i\sin\alpha\right)\left(\frac{x \pm iy}{r}\right)\right]$$

$$= e^{\mp i\alpha} Y_{1(\pm1)}(\hat{\mathbf{r}})$$

$$\Rightarrow M^1 = \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix} \qquad (m = 1, 0, -1)$$

More generally, it can be shown that:

$$M^l_{mm'} = e^{-im\alpha}\delta_{mm'}$$

Considering the spherical harmonic functions as basis functions of the three-dimensional rotation group, we can associate the matrix elements $M^l_{mm'}$ as irreducible representations. In this case, for each l, the dimension of the representation is $(2l+1)$.

$$D^l_{mm'}(\alpha) = e^{-im\alpha}\delta_{mm'} \qquad \text{for} \quad -l \le m, m' \le l$$

From this result, we can determine the characters of these representations:

$$\chi^l(\alpha) = \sum_{m=-l}^{l} e^{-im\alpha} = e^{-il\alpha}\frac{e^{i\alpha(2l+1)} - 1}{e^{i\alpha} - 1} = \frac{\sin\left[\left(l + \frac{1}{2}\right)\alpha\right]}{\sin\left(\frac{\alpha}{2}\right)}$$

Sanity check:

Note that the character of the identity class: $\chi^l(E) = 2l+1$

Now consider the great orthogonality theorem:

$$\sum_R \left(D^{\Gamma_n}(R)_{\mu\nu}\right)^* D^{\Gamma_{n'}}(R)_{\alpha\beta} = \frac{h}{l_n}\delta_{nn'}\delta_{\mu\alpha}\delta_{\nu\beta}$$

For continuous groups, the summation becomes an integral:

$$\int \left(D^{\Gamma_n}(R)_{\mu\nu}\right)^* D^{\Gamma_{n'}}(R)_{\alpha\beta}\, dR = \frac{\delta_{nn'}\delta_{\mu\alpha}\delta_{\nu\beta}}{l_n}\int dR$$

In terms of the characters of the representations:

$$\int \left(\chi^{\Gamma_n}(R)\right)^* \chi^{\Gamma_{n'}}(R)\, dR = \delta_{nn'}\int dR$$

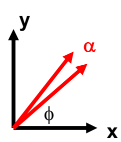
Procedure for carrying out integration over group elements

In general, there will be continuous parameter(s) which characterize each group element $R = R(\alpha, \beta, ...)$

$\int dR \Rightarrow \int g(R(\alpha,\beta...))\, d\alpha\, d\beta ...$ where $g(R(\alpha,\beta...))$ represents the density of group elements in the neighborhood of R in the parameter space of $\alpha, \beta ...$

Digression – "generators" of the three-dimensional rotation group

Consider rotation by an angle α about the z-axis



Let $\psi(\phi)$ denote the probability distribution for a quantum system in terms of its angle ϕ.

$$\langle\phi\rangle = \int d\phi\, \psi(\phi)^* \phi\, \psi(\phi)$$

$$\langle\phi\rangle + \alpha = \int d\phi\, \psi(\phi)^* (\phi + \alpha)\psi(\phi)$$

$$= \int d\phi'\, \psi(\phi' - \alpha)^* \phi'\, \psi(\phi' - \alpha)$$

Taylor expansion:

$$\psi(\phi'-\alpha)=\psi(\phi')-\alpha\frac{\partial\psi(\phi')}{\partial\phi'}+\frac{1}{2}\alpha^2\frac{\partial^2\psi(\phi')}{\partial\phi'^2}+....$$

$$=e^{-\alpha\frac{\partial}{\partial\phi'}}\psi(\phi')$$

$$=R_{-\alpha}\psi(\phi')$$

Generator operator for rotation: $=e^{-\alpha\frac{\partial}{\partial\phi'}}$