**PHY 712 Electrodynamics**
**9-9:50 AM MWF Olin 103**

**Plan for Lecture 26:**

**Start reading Chap. 11**

**A. Equations in cgs (Gaussian) units**

**B. Special theory of relativity**

**C. Lorentz transformation relations**



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|---|---|---|---|---|---|
| | Fri: 03/17/2017 | | *APS Meeting - no class* | | |
| 23 | Mon: 03/20/2017 | Chap. 9 | Sources of Radiation | #17 | 03/24/2017 |
| 24 | Wed: 03/22/2017 | Chap. 9 & 10 | Radiation and Scattering | | |
| 25 | Fri: 03/24/2017 | Chap. 9 & 10 | Radiation and Scattering | #18 | 03/27/2017 |
| 26 | Mon: 03/27/2017 | Chap. 11 | Special relativity | #19 | 03/31/2017 |
| 27 | Wed: 03/29/2017 | | | | |
| 28 | Fri: 03/30/2017 | | | | |
| 29 | Mon: 04/03/2017 | | | | |
| 30 | Wed: 04/05/2017 | | | | |
| 31 | Fri: 04/07/2017 | | | | |
| 32 | Mon: 04/10/2017 | | | | |
| 33 | Wed: 04/12/2017 | | | | |
| | Fri: 04/14/2017 | | *Good Friday Holiday -- no class* | | |
| 34 | Mon: 04/17/2017 | | | | |
| 35 | Wed: 04/19/2017 | | | | |
| 36 | Fri: 04/21/2017 | | | | |
| | Mon: 04/24/2017 | | Presentations I | | |
| | Wed: 04/26/2017 | | Presentations II | | |

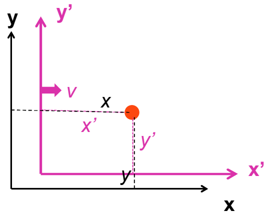


**Basic equations of electrodynamics**

| CGS (Gaussian) | SI |
|---|---|
| $\nabla \cdot \mathbf{D} = 4\pi\rho$ | $\nabla \cdot \mathbf{D} = \rho$ |
| $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot \mathbf{B} = 0$ |
| $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ | $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ |
| $\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$ | $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ |
| $\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$ | $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ |
| $u = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$ | $u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$ |
| $\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$ | $\mathbf{S} = (\mathbf{E} \times \mathbf{H})$ |

## Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum *c* is the same in all frames of reference.



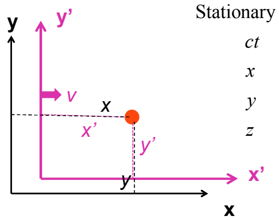



## Lorentz transformations

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$




| Stationary frame | | Moving frame |
|---|---|---|
| $ct$ | $=$ | $\gamma(ct'+\beta x')$ |
| $x$ | $=$ | $\gamma(x'+\beta ct')$ |
| $y$ | $=$ | $y'$ |
| $z$ | $=$ | $z'$ |



## Lorentz transformations -- continued

$$\beta \equiv \frac{v}{c} \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$ :

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice :

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

## Examples of other 4-vectors applicable to the Lorentz transformation:

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:  $\beta \equiv \frac{v}{c} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \qquad \text{Note: } \omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$$

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \qquad \text{Note: } E^2 - p^2c^2 = E'^2 - p'^2 c^2$$



## The Doppler Effect

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

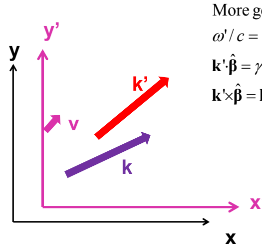
$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \qquad \text{Note: } \omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$$

$$\omega'/c = \gamma(\omega/c - \beta k_x) \qquad k'_x = \gamma(k_x - \beta\omega/c)$$
$$k'_y = k_y \qquad k'_z = k_z$$



## The Doppler Effect -- continued

$$\omega'/c = \gamma(\omega/c - \beta k_x) \qquad k'_x = \gamma(k_x - \beta\omega/c)$$
$$k'_y = k_y \qquad k'_z = k_z$$




More generally:

$$\omega'/c = \gamma(\omega/c - \boldsymbol{\beta}\cdot\mathbf{k}) \equiv \gamma(\omega/c - \beta k\cos\theta)$$
$$\mathbf{k}'\cdot\hat{\boldsymbol{\beta}} = \gamma(\mathbf{k}\cdot\hat{\boldsymbol{\beta}} - \beta\omega/c) \equiv k'\cos\theta' = \gamma(k\cos\theta - \beta\omega/c)$$
$$\mathbf{k}'\times\hat{\boldsymbol{\beta}} = \mathbf{k}\times\hat{\boldsymbol{\beta}}$$

For $\theta = 0$: $(k = \omega/c)$

$$\omega' = \omega\gamma(1-\beta) \qquad \Rightarrow \omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}$$

For $\theta \neq 0$: $(k = \omega/c)$

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

Electromagnetic Doppler Effect (θ=0)

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}} \qquad \beta \equiv \frac{v_{\text{source}} - v_{\text{detector}}}{c}$$

Sound Doppler Effect (θ=0)

$$\omega' = \omega\left(\frac{1 \pm v_{\text{detector}}/c_s}{1 \mp v_{\text{source}}/c_s}\right)$$



Lorentz transformation of the velocity

| Stationary frame | | Moving frame |
|---|---|---|
| $ct$ | = | $\gamma(ct'+\beta x')$ |
| $x$ | = | $\gamma(x'+\beta ct')$ |
| $y$ | = | $y'$ |
| $z$ | = | $z'$ |

For an infinitesimal increment:

| Stationary frame | | Moving frame |
|---|---|---|
| $cdt$ | = | $\gamma(cdt'+\beta dx')$ |
| $dx$ | = | $\gamma(dx'+\beta cdt')$ |
| $dy$ | = | $dy'$ |
| $dz$ | = | $dz'$ |



Lorentz transformation of the velocity -- continued

| Stationary frame | | Moving frame |
|---|---|---|
| $cdt$ | = | $\gamma(cdt'+\beta dx')$ |
| $dx$ | = | $\gamma(dx'+\beta cdt')$ |
| $dy$ | = | $dy'$ |
| $dz$ | = | $dz'$ |

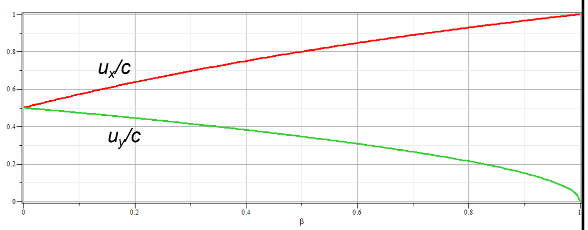
Define: $u_x \equiv \frac{dx}{dt} \quad u_y \equiv \frac{dy}{dt} \quad u_z \equiv \frac{dz}{dt}$

$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$

$$\frac{dx}{dt} = \frac{\gamma(dx'+\beta cdt')}{\gamma(dt'+\beta dx'/c)} = \frac{u'_x + v}{1+vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt'+\beta dx'/c)} = \frac{u'_y}{\gamma(1+vu'_x/c^2)} = u_y$$

Example of velocity variation with β:






## Extention to tranformation of acceleration

$$\mathbf{a}_{\|} = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\mathbf{a}'_{\|}$$

$$\mathbf{a}_{\perp} = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)^3}\left(\mathbf{a}'_{\perp}+\frac{\mathbf{v}}{c^2}\times\left(\mathbf{a}'\times\mathbf{u}'\right)\right)$$



Velocity transformations continued:

Consider: $u_x = \frac{u'_x + v}{1+vu'_x/c^2} \quad u_y = \frac{u'_y}{\gamma_v\left(1+vu'_x/c^2\right)} \quad u_z = \frac{u'_z}{\gamma_v\left(1+vu'_x/c^2\right)}.$

Note that $\gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1+vu'_x/c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}} = \gamma_v\gamma_{u'}\left(1+vu'_x/c^2\right)$

$\Rightarrow \gamma_u c = \gamma_v\left(\gamma_{u'}c + \beta_v\gamma_{u'}u'_x\right)$

$\Rightarrow \gamma_u u_x = \gamma_v\left(\gamma_{u'}u'_x + \gamma_{u'}v\right) = \gamma_v\left(\gamma_{u'}u'_x + \beta_v\gamma_{u'}c\right)$

$\Rightarrow \gamma_u u_y = \gamma_{u'}u'_y \qquad \gamma_u u_z = \gamma_{u'}u'_z$

Velocity 4-vector: $$\begin{pmatrix}\gamma_u c\\ \gamma_u u_x\\ \gamma_u u_y\\ \gamma_u u_z\end{pmatrix} = \mathcal{L}_u \begin{pmatrix}\gamma_{u'} c\\ \gamma_{u'} u'_x\\ \gamma_{u'} u'_y\\ \gamma_{u'} u'_z\end{pmatrix}$$

Some details:

$$\gamma_u = \gamma_v \gamma_{u'} \left(1 + vu'_x / c^2\right) \quad \Rightarrow \left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u'^2}{c^2}\right) = \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2$$

where $u_x = \dfrac{u'_x + v}{1 + vu'_x / c^2} \quad u_y = \dfrac{u'_y}{\gamma_v \left(1 + vu'_x / c^2\right)} \quad u_z = \dfrac{u'_z}{\gamma_v \left(1 + vu'_x / c^2\right)}.$

$$\left(\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2} + \frac{u_z^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2 = \left(\frac{u'_x}{c} + \frac{v}{c}\right)^2 + \left(\frac{u'^2_y}{c^2} + \frac{u'^2_z}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$

$$\frac{u^2}{c^2}\left(1 + \frac{u_x v}{c^2}\right)^2 = \frac{u'^2}{c^2}\left(1 - \frac{v^2}{c^2}\right) + \left(1 + \frac{u_x v}{c^2}\right)^2 - \left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \left(1 - \frac{u^2}{c^2}\right)\left(1 + \frac{u_x v}{c^2}\right)^2 = \left(1 - \frac{u'^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)$$



Significance of 4-velocity vector: $\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix}$

Introduce the "rest" mass *m* of particle characterized by velocity **u**:

$$mc\begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u mu_x c \\ \gamma_u mu_y c \\ \gamma_u mu_z c \end{pmatrix} = \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix}$$

Properties of energy-moment 4-vector:

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L}\begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{-1}\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \qquad \text{Note: } E^2 - p^2c^2 = E'^2 - p'^2 c^2$$



Properties of Energy-momentum 4-vector -- continued

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u mu_x c \\ \gamma_u mu_y c \\ \gamma_u mu_z c \end{pmatrix}$$

$$\text{Note: } E^2 - p^2c^2 = \frac{(mc^2)^2}{1 - \beta_u^2}\left(1 - \left(\frac{u_x}{c}\right)^2 - \left(\frac{u_y}{c}\right)^2 - \left(\frac{u_z}{c}\right)^2\right) = (mc^2)^2 = E'^2 - p'^2 c^2$$

Notion of "rest energy": For $\mathbf{p} \equiv 0, \quad E = mc^2$

Define kinetic energy: $E_K \equiv E - mc^2 = \sqrt{p^2c^2 + m^2c^4} - mc^2$

Non-relativistic limit: If $\dfrac{p}{mc} << 1, \quad E_K = mc^2\left(\sqrt{1 + \left(\dfrac{p}{mc}\right)^2} - 1\right)$

$$\approx \frac{p^2}{2m}$$

Summary of relativistic energy relationships

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \begin{pmatrix} \gamma_u mc^2 \\ \gamma_u m u_x c \\ \gamma_u m u_y c \\ \gamma_u m u_z c \end{pmatrix}$$

$$E = \sqrt{p^2c^2 + m^2c^4} = \gamma_u mc^2$$

$$\text{Check: } \sqrt{p^2c^2 + m^2c^4} = mc^2\sqrt{\gamma_u^2 \beta_u^2 + 1} = \gamma_u mc^2$$

Example: for an electron $mc^2 = 0.5\,\text{MeV}$

for $E = 200\,\text{GeV}$

$$\gamma_u = \frac{E}{mc^2} = 4\times 10^5$$

$$\beta_u = \sqrt{1-\frac{1}{\gamma_u^2}} \approx 1 - \frac{1}{2\gamma_u^2} \approx 1 - 3\times 10^{-12}$$



Special theory of relativity and Maxwell's equations

Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$

Lorentz gauge condition: $\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla\cdot\mathbf{A} = 0$

Potential equations: $\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} + \nabla^2\Phi = 4\pi\rho$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2\mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

Field relations: $\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$

$$\mathbf{B} = \nabla\times\mathbf{A}$$



More 4-vectors:

Time and position: $\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^\alpha$

Charge and current: $\begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^\alpha$

Vector and scalar potentials: $\begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^\alpha$

Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Time and space: $x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} x'^\beta$

Charge and current: $J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} J'^\beta$

Vector and scalar potential: $A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v^{\alpha\beta} A'^\beta$