

**PHY 712 Electrodynamics**  
**9-9:50 AM MWF Olin 103**

**Plan for Lecture 25:**

**Complete reading of Chap. 9 & 10**

**A. Superposition of radiation**

**B. Scattered radiation**

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Fri: 03/17/2017		APS Meeting - no class		
23 Mon: 03/20/2017	Chap. 9	Sources of Radiation	<a href="#">#17</a>	03/24/2017
24 Wed: 03/22/2017	Chap. 9 & 10	Radiation and Scattering		
25 Fri: 03/24/2017	Chap. 9 & 10	Radiation and Scattering	<a href="#">#18</a>	03/27/2017
26 Mon: 03/27/2017				
27 Wed: 03/29/2017				
28 Fri: 03/30/2017				
29 Mon: 04/03/2017				
30 Wed: 04/05/2017				
31 Fri: 04/07/2017				
32 Mon: 04/10/2017				
33 Wed: 04/12/2017				
Fri: 04/14/2017		Good Friday Holiday -- no class		
34 Mon: 04/17/2017				
35 Wed: 04/19/2017				
36 Fri: 04/21/2017				
Mon: 04/24/2017		Presentations I		
Wed: 04/26/2017		Presentations II		

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Electromagnetic waves from time harmonic sources – review:

For scalar potential (Lorentz gauge,  $k \equiv \frac{\omega}{c}$ )

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

For vector potential (Lorentz gauge,  $k \equiv \frac{\omega}{c}$ )

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

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Consider antenna source (center-fed)  
 Note – these notes differ from previous formulation  $d/2 \leftrightarrow d$

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} I \sin(k(d - |z|)) \delta(x) \delta(y) \quad \text{for } -d \leq z \leq d$$

$$k \equiv \frac{\omega}{c}$$

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Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} I \sin(k(d - |z|)) \delta(x) \delta(y) \quad \text{for } -d \leq z \leq d$$

$$k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \quad n = 1, 2, 3, \dots$$

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Consider antenna source -- continued

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} I \sin(k(d - |z|)) \delta(x) \delta(y) \quad \text{for } -d \leq z \leq d$$

$$k \equiv \frac{\omega}{c}$$

Vector potential from source:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

For  $r \gg d$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3 r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} I \int_{-d}^d dz' e^{-ikz' \cos\theta} \sin(k(d - |z'|))$$

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Consider antenna source -- continued

$$\begin{aligned} \tilde{\mathbf{A}}(\mathbf{r}, \omega) &\approx \hat{\mathbf{z}} \frac{\mu_0}{4\pi r} e^{ikr} \int_{-d}^d dz e^{-ikz \cos \theta} \sin(k(d-|z|)) \\ &= \hat{\mathbf{z}} \frac{\mu_0}{4\pi kr} e^{ikr} 2I \left[ \frac{\cos(kd \cos \theta) - \cos(kd)}{\sin^2 \theta} \right] \end{aligned}$$

In the radiation zone :

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx ik \hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) \approx -ikc \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r}, \omega))$$

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega)) = \frac{k^2 c}{2\mu_0} r^2 (|\tilde{\mathbf{A}}(\mathbf{r}, \omega)|^2 - |\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r}, \omega)|^2)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[ \frac{\cos(kd \cos \theta) - \cos(kd)}{\sin \theta} \right]^2$$

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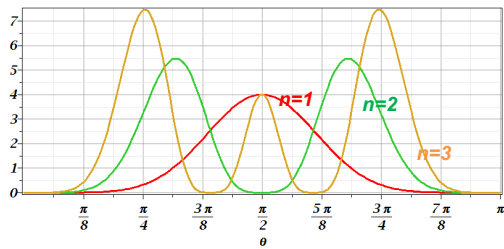
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Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[ \frac{\cos(kd \cos \theta) - \cos(kd)}{\sin \theta} \right]^2$$



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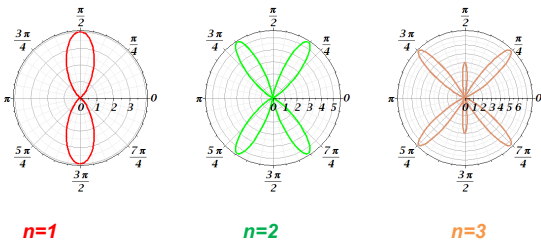
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Consider antenna source -- continued

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[ \frac{\cos(kd \cos \theta) - \cos(kd)}{\sin \theta} \right]^2$$

For  $kd = n\pi$ :



n=1

n=2

n=3

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Radiation from antenna arrays

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} I \sin(k(d-|z|)) \sum_{j=1}^{2N+1} \delta(x-(N+1-j)a) \delta(y) \quad \text{for } -d \leq z \leq d$$

$$k \equiv \frac{\omega}{c} = \frac{n\pi}{d}; \quad n = 1, 2, 3, \dots$$

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Radiation from antenna arrays -- continued

Vector potential from array source :

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega) \approx \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3 r' e^{-ik\hat{\mathbf{r}} \cdot \mathbf{r}'} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} I \sin(k(d-|z|)) \sum_{j=1}^{2N+1} \delta(x-(N+1-j)a) \delta(y) \quad \text{for } -d \leq z \leq d$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left( \sum_{j=-N}^N e^{-ikaj \sin \theta \cos \phi} \right) I \int_{-d}^d dz e^{-ikz \cos \theta} \sin(k(d-|z|))$$

$$\sum_{j=-N}^N e^{-ikaj \sin \theta \cos \phi} = \frac{\sin(\frac{1}{2}ka(2N+1)\sin \theta \cos \phi)}{\sin(\frac{1}{2}ka \sin \theta \cos \phi)}$$

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Radiation from antenna arrays -- continued

In the radiation zone :

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega) \approx ik\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) \approx -ikc\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \tilde{\mathbf{A}}(\mathbf{r}, \omega))$$

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0} r^2 \hat{\mathbf{r}} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega)) = \frac{k^2 cr^2}{2\mu_0} \left( |\tilde{\mathbf{A}}(\mathbf{r}, \omega)|^2 - |\hat{\mathbf{r}} \cdot \tilde{\mathbf{A}}(\mathbf{r}, \omega)|^2 \right)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[ \frac{\cos(kd \cos \theta) - \cos(kd)}{\sin \theta} \right]^2 \left[ \frac{\sin(\frac{1}{2}ka(2N+1)\sin \theta \cos \phi)}{\sin(\frac{1}{2}ka \sin \theta \cos \phi)} \right]^2$$

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$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{8\pi^2} I^2 \left[ \frac{\cos(kd \cos\theta) - \cos(kd)}{\sin\theta} \right]^2 \left[ \frac{\sin(\frac{1}{2}ka(2N+1)\sin\theta \cos\phi)}{\sin(\frac{1}{2}ka \sin\theta \cos\phi)} \right]^2$$

Example for  $\phi = 0, N = 10, kd = \pi = 2ka$

Additional amplitude patterns can be obtained by controlling relative phases of antennas.

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Brief introduction to multipole expansion of electromagnetic fields (Chap. 9.7)

Sourceless Maxwell's equations in terms of  $\mathbf{E}$  and  $\mathbf{H}$  fields with time dependence  $e^{-i\omega t}$ :

$$\nabla \times \mathbf{E} = ikZ_0 \mathbf{H} \quad \nabla \times \mathbf{H} = -ik\mathbf{E} / Z_0$$

$$\nabla \cdot \mathbf{E} = 0 \quad \nabla \cdot \mathbf{H} = 0$$

where  $k \equiv \omega / c$  and  $Z_0 \equiv \sqrt{\mu_0 / \epsilon_0}$

Decoupled equations:

$$(\nabla^2 + k^2)\mathbf{E} = 0 \quad (\nabla^2 + k^2)\mathbf{H} = 0$$

$$\mathbf{H} = -\frac{i}{kZ_0} \nabla \times \mathbf{E} \quad \mathbf{E} = \frac{iZ_0}{k} \nabla \times \mathbf{H}$$

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Multipole expansion of electromagnetic fields -- continued

Note that:

$$(\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{E}) = 0 \quad (\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{H}) = 0$$

Convenient operators for angular momentum analysis

Define:  $\mathbf{L} \equiv \frac{1}{i}(\mathbf{r} \times \nabla)$

Note that  $\mathbf{r} \cdot \mathbf{L} = 0$

$$\nabla^2 = \frac{1}{r} \frac{\partial^2 r}{\partial r^2} - \frac{L^2}{r^2}$$

Eigenfunctions:

$$L^2 Y_{lm}(\theta, \phi) = -\left[ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] Y_{lm}(\theta, \phi) = l(l+1) Y_{lm}(\theta, \phi)$$

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Multipole expansion of electromagnetic fields -- continued

Magnetic multipole field:

$$\mathbf{r} \cdot \mathbf{H}_{lm}^M \equiv \frac{l(l+1)}{k} g_l(kr) Y_{lm}(\theta, \phi)$$

$$\mathbf{r} \cdot \mathbf{E}_{lm}^M = 0$$

$$\mathbf{L} \cdot \mathbf{E}_{lm}^M = l(l+1) Z_0 g_l(kr) Y_{lm}(\theta, \phi)$$

Electric multipole field:

$$\mathbf{r} \cdot \mathbf{E}_{lm}^E \equiv -Z_0 \frac{l(l+1)}{k} f_l(kr) Y_{lm}(\theta, \phi)$$

$$\mathbf{r} \cdot \mathbf{H}_{lm}^E = 0$$

$$\mathbf{L} \cdot \mathbf{H}_{lm}^E = l(l+1) f_l(kr) Y_{lm}(\theta, \phi)$$

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Multipole expansion of electromagnetic fields -- continued

Vector spherical harmonics: (for  $l > 0$ )

$$\mathbf{X}_{lm}(\theta, \phi) = \frac{1}{\sqrt{l(l+1)}} \mathbf{L} Y_{lm}(\theta, \phi)$$

Orthogonality conditions:

$$\int d\Omega \mathbf{X}_{lm}^*(\theta, \phi) \cdot \mathbf{X}_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}$$

$$\int d\Omega \mathbf{X}_{lm}^*(\theta, \phi) \cdot (\mathbf{r} \times \mathbf{X}_{l'm'}(\theta, \phi)) = 0$$

General expansion of fields:

$$\mathbf{H} = \sum_{lm} \left[ a_{lm}^E f_l(kr) \mathbf{X}_{lm}(\theta, \phi) - \frac{i}{k} a_{lm}^M \nabla \times (g_l(kr) \mathbf{X}_{lm}(\theta, \phi)) \right]$$

$$\mathbf{E} = \sum_{lm} \left[ \frac{i}{k} a_{lm}^E \nabla \times (f_l(kr) \mathbf{X}_{lm}(\theta, \phi)) + a_{lm}^M g_l(kr) \mathbf{X}_{lm}(\theta, \phi) \right]$$

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Multipole expansion of electromagnetic fields -- continued

Time averaged power distribution of radiation far from source:

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \left| \sum_{lm} (-i)^{l+1} [a_{lm}^E \mathbf{X}_{lm}(\theta, \phi) \times \hat{\mathbf{r}} + a_{lm}^M \mathbf{X}_{lm}(\theta, \phi)] \right|^2$$

For a pure multipole radiation with either  $a_{lm}^E$  or  $a_{lm}^M$ :

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} |a_{lm}|^2 |\mathbf{X}_{lm}(\theta, \phi)|^2$$

$$|\mathbf{X}_{lm}(\theta, \phi)|^2 = \frac{1}{2l(l+1)} \left( 2m^2 |Y_{lm}|^2 + (l+m)(l-m+1) |Y_{l(m-1)}|^2 + (l-m)(l+m+1) |Y_{l(m+1)}|^2 \right)$$

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For example:  $l = 1$

$$|X_{10}(\theta, \phi)|^2 = \frac{3}{8\pi} \sin^2 \theta \quad |X_{11}(\theta, \phi)|^2 = |X_{1-1}(\theta, \phi)|^2 = \frac{3}{16\pi} (1 + \cos^2 \theta)$$

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For example:  $l = 2$

$$|X_{20}(\theta, \phi)|^2 = \frac{15}{8\pi} \sin^2 \theta \cos^2 \theta \quad |X_{21}(\theta, \phi)|^2 = \frac{5}{16\pi} (1 - 3\cos^2 \theta + 4\cos^4 \theta) \quad |X_{22}(\theta, \phi)|^2 = \frac{5}{16\pi} (1 - \cos^4 \theta)$$

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