

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 23:
Sources of radiation

Start reading Chap. 9

A. Electromagnetic waves due to specific sources

B. Dipole radiation patterns

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Mon: 03/13/2017		APS Meeting - no class		
Wed: 03/15/2017		APS Meeting - no class		
Fri: 03/17/2017		APS Meeting - no class		
23 Mon: 03/20/2017	Chap. 9	Sources of Radiation	#17	03/24/2017
24 Wed: 03/22/2017				
25 Fri: 03/24/2017				
26 Mon: 03/27/2017				
27 Wed: 03/29/2017				
28 Fri: 03/30/2017				
29 Mon: 04/03/2017				
30 Wed: 04/05/2017				
31 Fri: 04/07/2017				
32 Mon: 04/10/2017				
33 Wed: 04/12/2017				
Fri: 04/14/2017		Good Friday Holiday - no class		
34 Mon: 04/17/2017				
35 Wed: 04/19/2017				
36 Fri: 04/21/2017				
Mon: 04/24/2017		Presentations I		
Wed: 04/26/2017		Presentations II		

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Comment on Mid-term exam

$$\mathbf{k}_{0I} = \frac{n\omega}{c}(\sin i \hat{x} + \cos i \hat{y})$$

$$\mathbf{E}_{0I} = \frac{E_{0I}}{\sqrt{2}}(-\cos i \hat{x} + \sin i \hat{y} + \hat{z})$$

$$\mathbf{k}_{0R} = \frac{n\omega}{c}(\sin i \hat{x} - \cos i \hat{y})$$

$$\mathbf{E}_{0R} = E_{0R}^s \hat{z} + E_{0R}^p (\cos i \hat{x} + \sin i \hat{y})$$

$$\mathbf{k}_{0T} = \frac{n'\omega}{c}(\sin \theta \hat{x} + \cos \theta \hat{y})$$

$$\mathbf{E}_{0T} = E_{0T}^s \hat{z} + E_{0T}^p (-\cos \theta \hat{x} + \sin \theta \hat{y})$$

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Maxwell's equations

Microscopic or vacuum form ($\mathbf{P} = 0$; $\mathbf{M} = 0$):

Coulomb's law : $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

or $\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require : $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

General equation form :

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi = -4\pi f$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right)$$

Solution for field $\Psi(\mathbf{r}, t)$:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) f(\mathbf{r}', t')$$

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Electromagnetic waves from time harmonic sources

Charge density : $\rho(\mathbf{r}, t) = \Re(\tilde{\rho}(\mathbf{r}, \omega)e^{-i\omega t})$

Current density : $\mathbf{J}(\mathbf{r}, t) = \Re(\tilde{\mathbf{J}}(\mathbf{r}, \omega)e^{-i\omega t})$

Note that the continuity condition :

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) = 0 \Rightarrow -i\omega \tilde{\rho}(\mathbf{r}, \omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = 0$$

General source : $f(\mathbf{r}, t) = \Re(\tilde{f}(\mathbf{r}, \omega)e^{-i\omega t})$

For $\tilde{f}(\mathbf{r}, \omega) = \frac{1}{4\pi\epsilon_0} \tilde{\rho}(\mathbf{r}, \omega)$

or $\tilde{f}(\mathbf{r}, \omega) = \frac{\mu_0}{4\pi} \tilde{\mathbf{J}}_i(\mathbf{r}, \omega)$

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Electromagnetic waves from time harmonic sources -- continued:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) f(\mathbf{r}', t')$$

$$\tilde{\Psi}(\mathbf{r}, \omega)e^{-i\omega t} = \tilde{\Psi}_{f=0}(\mathbf{r}, \omega)e^{-i\omega t} +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|\right)\right) \tilde{f}(\mathbf{r}', \omega)e^{-i\omega t'}$$

$$= \tilde{\Psi}_{f=0}(\mathbf{r}, \omega)e^{-i\omega t} + \int d^3r' \frac{e^{i\frac{\omega}{c}|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \tilde{f}(\mathbf{r}', \omega)e^{-i\omega t}$$

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Electromagnetic waves from time harmonic sources – continued:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\rho}(\mathbf{r}', \omega)$$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \frac{\mu_0}{4\pi} \int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tilde{\mathbf{J}}(\mathbf{r}', \omega)$$

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Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

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Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

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Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x} \qquad h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad h_1(x) = -\left(1 + \frac{i}{x}\right) \frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin(x) - \frac{3\cos(x)}{x^2} \qquad h_2(x) = i\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right) \frac{e^{ix}}{x}$$

Asymptotic behavior :

$$x \ll 1 \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!}$$

$$x \gg 1 \Rightarrow h_l(x) \approx (-i)^{l+1} \frac{e^{ix}}{x}$$

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Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{\Phi}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\phi}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r, \omega) = \frac{ik}{\epsilon_0} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \tilde{\mathbf{A}}_0(\mathbf{r}, \omega) + \sum_{lm} \tilde{\mathbf{a}}_{lm}(r, \omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) = ik\mu_0 \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr_<) h_l(kr_>) Y_{lm}^*(\hat{\mathbf{r}}')$$

For $r \gg$ (extent of source)

$$\tilde{\phi}_{lm}(r, \omega) \approx \frac{ik}{\epsilon_0} h_l(kr) \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) \approx ik\mu_0 h_l(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

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Electromagnetic waves from time harmonic sources – continued:

For $r \gg$ (extent of source)

$$\tilde{\phi}_{lm}(r, \omega) \approx \frac{ik}{\epsilon_0} h_l(kr) \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{lm}(r, \omega) \approx ik\mu_0 h_l(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that $\tilde{\rho}(\mathbf{r}', \omega)$ and $\tilde{\mathbf{J}}(\mathbf{r}', \omega)$ are connected via the continuity condition : $-i\omega \tilde{\rho}(\mathbf{r}, \omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r}, \omega) = 0$

$$\tilde{\phi}_{lm}(r, \omega) \approx \frac{ik}{\epsilon_0} h_l(kr) \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}')$$

$$= -\frac{k}{\omega\epsilon_0} h_l(kr) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) \cdot \nabla' (j_l(kr') Y_{lm}^*(\hat{\mathbf{r}}'))$$

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Electromagnetic waves from time harmonic sources – continued:

Various approximations :

$$kr \gg 1 \Rightarrow h_l(kr) \approx (-i)^{l+1} \frac{e^{ikr}}{kr}$$

$$kr \ll 1 \Rightarrow j_l(kr) \approx \frac{(kr)^l}{(2l+1)!}$$

Lowest (non - trivial) contributions in l expansions :

$$\tilde{\phi}_m(r, \omega) \approx -\frac{ik}{\epsilon_0} \frac{e^{ikr}}{kr} \int d^3r' \tilde{\rho}(\mathbf{r}', \omega) \frac{kr'}{3} Y_{lm}^*(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{a}}_{00}(r, \omega) \approx ik\mu_0 \left(-i \frac{e^{ikr}}{kr} \right) \int d^3r' \tilde{\mathbf{J}}(\mathbf{r}', \omega) Y_{00}^*(\hat{\mathbf{r}}')$$

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Electromagnetic waves from time harmonic sources – continued:

Lowest order contribution; dipole radiation :

Define dipole moment at frequency ω :

$$\mathbf{p}(\omega) \equiv \int d^3r \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \tilde{\mathbf{J}}(\mathbf{r}, \omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{i}{4\pi\omega\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr} \right) \frac{e^{ikr}}{r}$$

Note: in this case we have assumed a restricted extent of the source such that $kr \ll 1$.

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Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = -\nabla\tilde{\Phi}(\mathbf{r}, \omega) + i\omega\tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{e^{ikr}}{r} \left(k^2((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}) + \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2} \right) (1 - ikr) \right)$$

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \nabla \times \tilde{\mathbf{A}}(\mathbf{r}, \omega)$$

$$= \frac{1}{4\pi\epsilon_0 c^2} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \left(1 - \frac{1}{ikr} \right)$$

Power radiated for $kr \gg 1$:

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{r^2 \hat{\mathbf{r}}}{2\mu_0} \cdot \Re(\tilde{\mathbf{E}}(\mathbf{r}, \omega) \times \tilde{\mathbf{B}}^*(\mathbf{r}, \omega))$$

$$= \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\epsilon_0}} |(\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}}|^2$$

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Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 e^{-r/R} \quad \tilde{\rho}(\mathbf{r}, \omega) = \frac{J_0}{-i\omega R} \cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 (ik\mu_0) \int_0^\infty r'^2 dr' e^{-r'/R} h_0(kr'_z) j_0(kr'_z)$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{J_0 k}{\epsilon_0 \omega R} \cos\theta \int_0^\infty r'^2 dr' e^{-r'/R} h_1(kr'_z) j_1(kr'_z)$$

Evaluation for $r \gg R$:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{(1+k^2R^2)^2}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{J_0 k}{\epsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \frac{2R^3}{(1+k^2R^2)^2}$$

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Example of dipole radiation source -- continued

Evaluation for $r \gg R$:

$$\tilde{\mathbf{A}}(\mathbf{r}, \omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{(1+k^2R^2)^2}$$

$$\tilde{\Phi}(\mathbf{r}, \omega) = \frac{J_0 k}{\epsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \frac{2R^3}{(1+k^2R^2)^2}$$

Relationship to pure dipole approximation (exact when $kR \rightarrow 0$)

$$\mathbf{p}(\omega) \equiv \int d^3r \mathbf{r} \tilde{\rho}(\mathbf{r}, \omega) = -\frac{1}{i\omega} \int d^3r \tilde{\mathbf{J}}(\mathbf{r}, \omega) = -\frac{8\pi R^3 J_0}{i\omega} \hat{\mathbf{z}}$$

Corresponding dipole fields: $\tilde{\mathbf{A}}(\mathbf{r}, \omega) = -\frac{i\mu_0 \omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r}$

$$\tilde{\Phi}(\mathbf{r}, \omega) = -\frac{i}{4\pi\omega\epsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr}\right) \frac{e^{ikr}}{r}$$

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