PHY 712 Electrodynamics
9-9:50 AM Olin 103

Plan for Lecture 22:

Chap. 8 in Jackson – Wave Guides

1. **Electromagnetic waves within an ideal rectangular wave guide**
2. **Electromagnetic waves within an ideal cylindrical wave guide**
3. **Energy considerations**

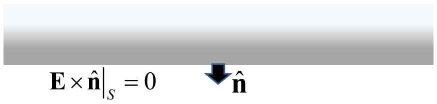
12	Wed: 02/08/2017	Chap. 5	Magnetostatics and the Hyperfine Interaction	#12	02/10/2017
13	Fri: 02/10/2017	Chap. 5	Magnetic dipoles and dipolar fields	#13	02/13/2017
14	Mon: 02/13/2017	Chap. 6	Maxwell's Equations	#14	02/15/2017
15	Wed: 02/15/2017	Chap. 6	Electromagnetic energy and forces	#15	02/17/2017
16	Fri: 02/17/2017	Chap. 7	Electromagnetic plane waves	#16	02/20/2017
17	Mon: 02/20/2017	Chap. 7	Dielectric media		
18	Wed: 02/22/2017	Chap. 7	Complex dielectrics		
19	Fri: 02/24/2017	Chap. 1-7	Review -- Take home exam distributed		
20	Mon: 02/27/2017	Chap. 8	Wave guides		Exam
21	Wed: 03/01/2017	Chap. 8	Wave guides		Exam
22	Fri: 03/03/2017	Chap. 8	Wave guides		Exam Due
	Mon: 03/06/2017		*Spring break - no class*		
	Wed: 03/08/2017		*Spring break - no class*		
	Fri: 03/10/2017		*Spring break - no class*		
	Mon: 03/13/2017		*APS Meeting - no class*		
	Wed: 03/15/2017		*APS Meeting - no class*		
	Fri: 03/17/2017		*APS Meeting - no class*		
23	Mon: 03/20/2017				
24	Wed: 03/22/2017				

Comments:

- Enjoy your spring break!
- Please use part of the week of 3/13-17 to develop your presentation projects

	Fri: 03/17/2017		*APS Meeting - no class*		
23	Mon: 03/20/2017				
24	Wed: 03/22/2017				
25	Fri: 03/24/2017				
26	Mon: 03/27/2017				
27	Wed: 03/29/2017				
28	Fri: 03/30/2017				
29	Mon: 04/03/2017				
30	Wed: 04/05/2017				
31	Fri: 04/07/2017				
32	Mon: 04/10/2017				
33	Wed: 04/12/2017				
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017				
35	Wed: 04/19/2017				
36	Fri: 04/21/2017				
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		

Boundary conditions at surface of ideal metal



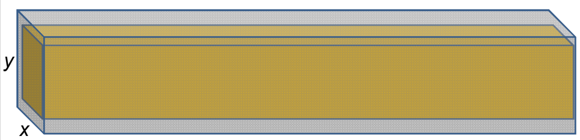
$$\mathbf{E} \times \hat{\mathbf{n}}\big|_S = 0$$

$$\mathbf{H} \cdot \hat{\mathbf{n}}\big|_S = 0$$

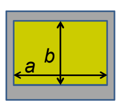
Analysis of rectangular waveguide

Boundary conditions at surface of waveguide:

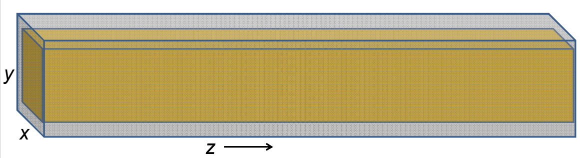
$$\mathbf{E} \times \hat{\mathbf{n}}\big|_S = 0 \qquad \mathbf{H} \cdot \hat{\mathbf{n}}\big|_S = 0$$



Cross section view



Analysis of rectangular waveguide



$$\mathbf{H} = \Re\left\{\left(H_x(x,y)\hat{\mathbf{x}} + H_y(x,y)\hat{\mathbf{y}} + H_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Inside the dielectric medium:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} + \mu \frac{\partial \mathbf{H}}{\partial t} = 0 \qquad \nabla \times \mathbf{H} - \varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$$

Solution of Maxwell's equations within the pipe:

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholz equation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-k^2+\mu\varepsilon\omega^2\right)E_x(x,y) \;=\; 0.$$

For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad H_z(x,y)=H_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$\text{with } k^2\equiv k_{mn}^2=\mu\varepsilon\omega^2-\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]$$

Maxwell's equations within the pipe in terms of all 6 components:

For TE mode

$$\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+ikH_z=0.$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+ik\cancel{E_z}=0.$$

$$\frac{\partial E_z}{\partial y}-ikE_y=i\omega\mu H_x. \qquad \frac{\partial H_z}{\partial y}-ikH_y=-i\varepsilon\omega E_x.$$

$$ikE_x-\cancel{\frac{\partial E_z}{\partial x}}=i\omega\mu H_y. \qquad ikH_x-\frac{\partial H_z}{\partial x}=-i\varepsilon\omega E_y.$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega\mu H_z. \qquad \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=-i\varepsilon\omega\cancel{E_z}.$$

TE modes for retangular wave guide continued:

$$E_z(x,y)\equiv 0 \quad \text{and} \quad H_z(x,y)=H_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right),$$

$$E_x=\frac{\omega\mu}{k}H_y=\frac{-i\omega\mu}{k^2-\mu\varepsilon\omega^2}\frac{\partial B_z}{\partial y}=\frac{-i\omega\mu}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{n\pi}{b}H_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

$$E_y=-\frac{\omega\mu}{k}H_x=\frac{i\omega\mu}{k^2-\mu\varepsilon\omega^2}\frac{\partial H_z}{\partial x}=\frac{i\omega\mu}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\frac{m\pi}{a}H_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Check boundary conditions:

$$\mathbf{E}\times\mathbf{n}\big|_S=0 \text{ because: } \quad E_x(x,0)=E_x(x,b)=0$$

$$\text{and } E_y(0,y)=E_y(a,y) \;=\; 0.$$

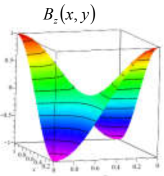
Solution for m=n=1

$$B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

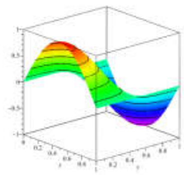
$$iE_x(x,y) = B_0\left(\frac{\omega n\pi / b}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$iE_y(x,y) = B_0\left(\frac{-\omega m\pi / a}{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

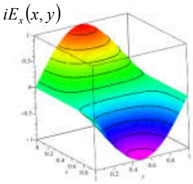
$B_z(x,y)$



$iE_y(x,y)$

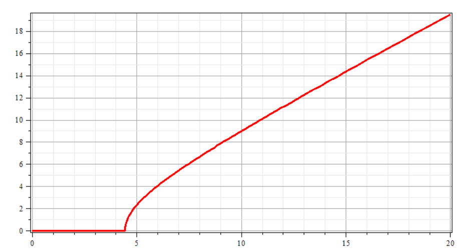


$iE_x(x,y)$

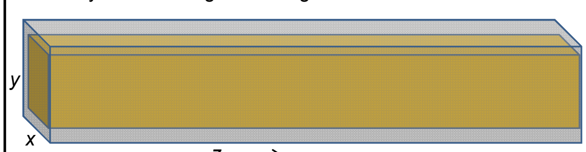


Solution for m=n=1

$$k^2 \equiv k_{mn}^2 = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$



Analysis of rectangular waveguide



$$\mathbf{H} = \Re\left\{\left(H_x(x,y)\hat{\mathbf{x}} + H_y(x,y)\hat{\mathbf{y}} + H_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(E_x(x,y)\hat{\mathbf{x}} + E_y(x,y)\hat{\mathbf{y}} + E_z(x,y)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Simplified notation: $(x,y) \rightarrow \mathbf{r}_\perp$

$$\mathbf{H} = \Re\left\{\left(\mathbf{H}_\perp(\mathbf{r}_\perp) + H_z(\mathbf{r}_\perp)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(\mathbf{E}_\perp(\mathbf{r}_\perp) + E_z(\mathbf{r}_\perp)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{H} = \Re\left\{\left(\mathbf{H}_\perp(\mathbf{r}_\perp) + H_z(\mathbf{r}_\perp)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

$$\mathbf{E} = \Re\left\{\left(\mathbf{E}_\perp(\mathbf{r}_\perp) + E_z(\mathbf{r}_\perp)\hat{\mathbf{z}}\right)e^{ikz-i\omega t}\right\}$$

Maxwell's Equations inside dielectric:

$$\nabla\times\mathbf{E} = i\omega\mu\mathbf{H} \qquad \nabla\times\mathbf{H} = -i\omega\varepsilon\mathbf{E}$$

$$\nabla\times\mathbf{E} = \Re\left\{\left(\nabla_\perp\times\mathbf{E}_\perp(\mathbf{r}_\perp) + \nabla_\perp E_z(\mathbf{r}_\perp)\times\hat{\mathbf{z}} + ik\hat{\mathbf{z}}\times\mathbf{E}_\perp(\mathbf{r}_\perp)\right)e^{ikz-i\omega t}\right\}$$

$$\Rightarrow \nabla_\perp\times\mathbf{E}_\perp(\mathbf{r}_\perp) = i\omega\mu H_z(\mathbf{r}_\perp)\hat{\mathbf{z}}$$

$$\nabla_\perp E_z(\mathbf{r}_\perp)\times\hat{\mathbf{z}} + ik\hat{\mathbf{z}}\times\mathbf{E}_\perp(\mathbf{r}_\perp) = i\omega\mu\mathbf{H}_\perp(\mathbf{r}_\perp)$$

Similarly,

$$\nabla_\perp\times\mathbf{H}_\perp(\mathbf{r}_\perp) = -i\omega\varepsilon E_z(\mathbf{r}_\perp)\hat{\mathbf{z}}$$

$$\nabla_\perp H_z(\mathbf{r}_\perp)\times\hat{\mathbf{z}} + ik\hat{\mathbf{z}}\times\mathbf{H}_\perp(\mathbf{r}_\perp) = -i\omega\varepsilon\mathbf{E}_\perp(\mathbf{r}_\perp)$$

03/03/2017 PHY 712 Spring 2017 -- Lecture 22 13

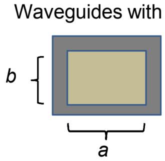
Note that all of the fields $\mathbf{H}_\perp(\mathbf{r}_\perp), H_z(\mathbf{r}_\perp), \mathbf{E}_\perp(\mathbf{r}_\perp)$, and $E_z(\mathbf{r}_\perp)$ must satisfy the equation:

$$\left(\nabla_\perp^2 + \gamma^2\right)E_z(\mathbf{r}_\perp) = 0, \text{ etc. } \quad \text{for } \gamma^2 \equiv \mu\varepsilon\omega^2 - k^2$$

Transverse electric solution (TE)	Transverse magnetic solution (TM)
$E_z(\mathbf{r}_\perp) = 0$	$H_z(\mathbf{r}_\perp) = 0$
$\left(\nabla_\perp^2 + \gamma^2\right)H_z(\mathbf{r}_\perp) = 0$	$\left(\nabla_\perp^2 + \gamma^2\right)E_z(\mathbf{r}_\perp) = 0$
$\mathbf{H}_\perp(\mathbf{r}_\perp) = \frac{ik}{\gamma^2}\nabla_\perp H_z(\mathbf{r}_\perp)$	$\mathbf{E}_\perp(\mathbf{r}_\perp) = \frac{ik}{\gamma^2}\nabla_\perp E_z(\mathbf{r}_\perp)$
$\mathbf{E}_\perp(\mathbf{r}_\perp) = -\frac{\omega\mu}{k}\hat{\mathbf{z}}\times\mathbf{H}_\perp(\mathbf{r}_\perp)$	$\mathbf{H}_\perp(\mathbf{r}_\perp) = -\frac{\omega\mu}{k}\hat{\mathbf{z}}\times\mathbf{E}_\perp(\mathbf{r}_\perp)$

03/03/2017 PHY 712 Spring 2017 -- Lecture 22 14

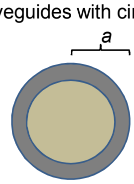
Waveguides with rectangular cross section



$$\text{TM:}\quad E_z(\mathbf{r}_\perp) = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$\text{TE:}\quad H_z(\mathbf{r}_\perp) = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)$$

Waveguides with circular cross section

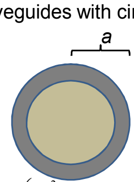


Boundary conditions:

$$\mathbf{E}_\perp(\mathbf{r}_\perp)\Big|_S = 0 \qquad \left.\frac{\partial\mathbf{H}_\perp(\mathbf{r}_\perp)}{\partial n}\right|_S = 0$$

03/03/2017 PHY 712 Spring 2017 -- Lecture 22 15

Waveguides with circular cross section



$$\text{TM}: \left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\gamma^2\right)E_z(\rho,\phi)=0 \qquad E_z(a,\phi)=0$$

$$\text{TE}: \left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\gamma^2\right)H_z(\rho,\phi)=0 \qquad \frac{\partial H_z(a,\phi)}{\partial n}=0$$

$$E_z(\rho,\phi)=E_0 J_m\left(\gamma^{TM}{}_{mn}\rho\right)e^{im\phi} \qquad H_z(\rho,\phi)=H_0 J_m\left(\gamma^{TE}{}_{mn}\rho\right)e^{im\phi}$$

$$J_m\left(\gamma^{TM}{}_{mn}a\right)=0 \qquad \frac{\partial J_m\left(\gamma^{TM}{}_{mn}a\right)}{\partial\rho}=0$$

Energy associated with wave guide modes

$$\langle P\rangle=\int d^2r_\perp \langle \mathbf{S}\rangle\cdot\hat{\mathbf{z}}=\begin{cases}\dfrac{\omega\mu k}{2\gamma^2}\displaystyle\int d^2r_\perp\left|H_z\right|^2 & \text{TE}\\ \dfrac{\omega\varepsilon k}{2\gamma^2}\displaystyle\int d^2r_\perp\left|E_z\right|^2 & \text{TM}\end{cases}$$

In practice, there are energy losses due conduction within the skin depth.

Comments:

- Enjoy your spring break!
- Please use part of the week of 3/13-17 to develop your presentation projects

	Fri: 03/17/2017		*APS Meeting - no class*		
23	Mon: 03/20/2017				
24	Wed: 03/22/2017				
25	Fri: 03/24/2017				
26	Mon: 03/27/2017				
27	Wed: 03/29/2017				
28	Fri: 03/30/2017				
29	Mon: 04/03/2017				
30	Wed: 04/05/2017				
31	Fri: 04/07/2017				
32	Mon: 04/10/2017				
33	Wed: 04/12/2017				
	Fri: 04/14/2017		*Good Friday Holiday -- no class*		
34	Mon: 04/17/2017				
35	Wed: 04/19/2017				
36	Fri: 04/21/2017				
	Mon: 04/24/2017		Presentations I		
	Wed: 04/26/2017		Presentations II		