PHY 712 Electrodynamics
9-9:50 AM Olin 103

Plan for Lecture 19:

Review Chapter 1-7 in Jackson

1. **Brief review**
2. **Comments on some homework problems**
3. **Distribution of take home exam**

10	Fri: 02/03/2017	Chap. 4	Dipoles and dielectrics	#10	02/06/2017
11	Mon: 02/06/2017	Chap. 5	Magnetostatics	#11	02/08/2017
12	Wed: 02/08/2017	Chap. 5	Magnetostatics and the Hyperfine Interaction	#12	02/10/2017
13	Fri: 02/10/2017	Chap. 5	Magnetic dipoles and dipolar fields	#13	02/13/2017
14	Mon: 02/13/2017	Chap. 6	Maxwell's Equations	#14	02/15/2017
15	Wed: 02/15/2017	Chap. 6	Electromagnetic energy and forces	#15	02/17/2017
16	Fri: 02/17/2017	Chap. 7	Electromagnetic plane waves	#16	02/20/2017
17	Mon: 02/20/2017	Chap. 7	Dielectric media		
18	Wed: 02/22/2017	Chap. 7	Complex dielectrics		
19	Fri: 02/24/2017	Chap. 1-7	Review -- Take home exam distributed		
20	Mon: 02/27/2017				Exam
21	Wed: 03/01/2017				Exam
22	Fri: 03/03/2017				Exam Due
	Mon: 03/06/2017		*Spring break - no class*		
	Wed: 03/08/2017		*Spring break - no class*		
	Fri: 03/10/2017		*Spring break - no class*		
	Mon: 03/13/2017		*APS Meeting - no class*		
	Wed: 03/15/2017		*APS Meeting - no class*		
	Fri: 03/17/2017		*APS Meeting - no class*		
23	Mon: 03/20/2017				
24	Wed: 03/22/2017				
25	Fri: 03/24/2017				

Review

Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

For linear isotropic media and no sources: $\mathbf{D} = \varepsilon \mathbf{E}$; $\mathbf{B} = \mu \mathbf{H}$

Review -- continued

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

Review -- continued

$$\nabla\cdot\mathbf{B}=0 \qquad \Rightarrow \mathbf{B}=\nabla\times\mathbf{A}$$

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \qquad \Rightarrow \nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)=0$$

$$\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}=-\nabla\Phi$$

$$\text{or}\quad \mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

Review -- continued

Analysis of the scalar and vector potential equations:

$$-\nabla^2\Phi-\frac{\partial(\nabla\cdot\mathbf{A})}{\partial t}=\rho/\varepsilon_0$$

$$\nabla\times(\nabla\times\mathbf{A})+\frac{1}{c^2}\left(\frac{\partial(\nabla\Phi)}{\partial t}+\frac{\partial^2\mathbf{A}}{\partial t^2}\right)=\mu_0\mathbf{J}$$

Lorentz gauge form -- require $\nabla\cdot\mathbf{A}_L+\frac{1}{c^2}\frac{\partial\Phi_L}{\partial t}=0$

$$-\nabla^2\Phi_L+\frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2}=\rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L+\frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2}=\mu_0\mathbf{J}$$

Review -- continued

When to solve equations using integral form versus differential form?

Examples from electrostatic and magnetostatic cases:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Useful for spatially confined sources.

Review -- continued

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi')$$

Review -- continued

Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2(\pm2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

$$Y_{2(\pm1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Review -- continued

General form of electrostatic potential with boundary value $r \rightarrow \infty$, for isolated charge density $\rho(\mathbf{r})$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')\right)$$

Suppose that $\rho(\mathbf{r}) = \sum_{lm}\rho_{lm}(r)Y_{lm}(\theta,\varphi)$

$$\Rightarrow \Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}dr'\,\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}dr'\rho_{lm}(r')\right)$$

Review -- continued

Hyperfine interaction energy:

$$E_{int} \equiv H_{HF} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$$

Putting all of the terms together:

$$H_{\mathrm{HF}} = -\frac{\mu_0}{4\pi}\left(\left\langle\frac{3(\boldsymbol{\mu}_{\mathrm{N}}\cdot\hat{\mathbf{r}})(\boldsymbol{\mu}_{\mathrm{e}}\cdot\hat{\mathbf{r}}) - \boldsymbol{\mu}_{\mathrm{N}}\cdot\boldsymbol{\mu}_{\mathrm{e}}}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_{\mathrm{N}}\cdot\boldsymbol{\mu}_{\mathrm{e}}\delta^3(\mathbf{r})\right\rangle + \frac{e}{m_e}\left\langle\frac{\mathbf{L}\cdot\boldsymbol{\mu}_{\mathrm{N}}}{r^3}\right\rangle\right).$$

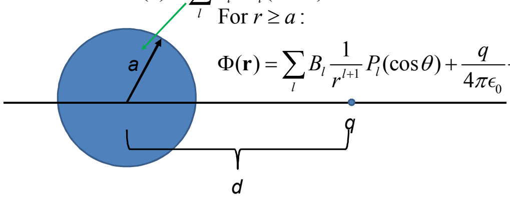
In this expression the brackets $\langle\ \rangle$ indicate evaluating the expectation value relative to the electronic state.

Review of HW problem:
4.9 in Jackson
A point charge q is located in free space a distance d from the center of a dielectric sphere of radius a ($a<d$) and dielectric constant $\varepsilon/\varepsilon_0$. Find the electrostatic potential. For $r \leq a$:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos\theta)$$

For $r \geq a$:

$$\Phi(\mathbf{r}) = \sum_l B_l \frac{1}{r^{l+1}}P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-d\hat{\mathbf{z}}|}$$



Review of HW -- continued

For $r \le a$:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos\theta)$$

For $r \ge a$:

$$\Phi(\mathbf{r}) = \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r} - d\hat{\mathbf{z}}|}$$

In order to match BC's at $r = a$:

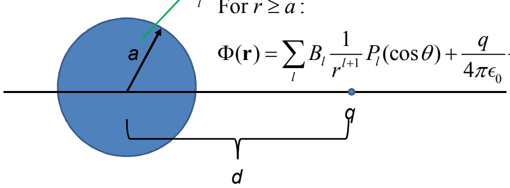
$$\frac{1}{|\mathbf{r} - d\hat{\mathbf{z}}|} = \sum_{l=0} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta) = \sum_{l=0} \frac{a^l}{d^{l+1}} P_l(\cos\theta)$$

For $r \le a$:

$$\Phi(\mathbf{r}) = \sum_l A_l r^l P_l(\cos\theta)$$

For $r \ge a$:

$$\Phi(\mathbf{r}) = \sum_l B_l \frac{1}{r^{l+1}} P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\frac{1}{|\mathbf{r} - d\hat{\mathbf{z}}|}$$



Boundary conditions:

$\mathbf{D}\cdot\hat{\mathbf{r}}\big|_{r=a}$ = continuous $\qquad$ $\mathbf{E}\cdot\hat{\boldsymbol{\theta}}\big|_{r=a}$ = continuous

$$\varepsilon \left.\frac{\partial\Phi_{in}(r)}{\partial r}\right|_{r=a} = \varepsilon_0 \left.\frac{\partial\Phi_{out}(r)}{\partial r}\right|_{r=a} \qquad \left.\frac{\partial\Phi_{in}(r)}{\partial\theta}\right|_{r=a} = \left.\frac{\partial\Phi_{out}(r)}{\partial\theta}\right|_{r=a}$$

Boundary conditions:

$\mathbf{D}\cdot\hat{\mathbf{r}}\big|_{r=a}$ = continuous $\qquad$ $\mathbf{E}\cdot\hat{\boldsymbol{\theta}}\big|_{r=a}$ = continuous

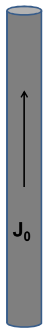
$$\varepsilon \left.\frac{\partial\Phi_{in}(r)}{\partial r}\right|_{r=a} = \varepsilon_0 \left.\frac{\partial\Phi_{out}(r)}{\partial r}\right|_{r=a} \qquad \left.\frac{\partial\Phi_{in}(r)}{\partial\theta}\right|_{r=a} = \left.\frac{\partial\Phi_{out}(r)}{\partial\theta}\right|_{r=a}$$

Equality for each l:

$$\varepsilon l a^{l-1} A_l = -\frac{\varepsilon_0 (l+1)}{a^{l+2}} B_l + \frac{q}{4\pi}\frac{l a^{l-1}}{d^{l+1}}$$

$$a^l A_l = +\frac{1}{a^{l+1}} B_l + \frac{q}{4\pi}\frac{a^l}{d^{l+1}}$$

Comment on HW #11

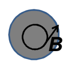


1. Consider an infinitely long wire with radius a, oriented along the **z** axis. There is a steady uniform current inside the wire. Specifically the current is along the z-axis with the magnitude of J_0 for $\rho \le a$ and zero for $\rho > a$, where ρ denotes the radial parameter of the natural cylindrical coordinates of the system.
   a. Find the vector potential (**A**) for all ρ.
   b. Find the magnetic flux field (**B**) for all ρ.

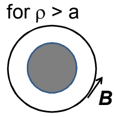
Solution to problem using PHY 114 ideas

In this case, it is convenient to solve part b first.

Top view for $\rho < a$

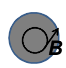


Top view for $\rho > a$



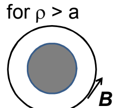
Comment on HW #11 -- continued

Top view for $\rho < a$



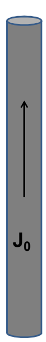
$$\oint \mathbf{B}\cdot\mathbf{d}\ell = \mu_0 \int \mathbf{J}\cdot\mathbf{dA}$$
$$2\pi\rho B = \mu_0 J_0 \pi\rho^2$$
$$B = \frac{\mu_0 J_0 \rho}{2}$$
$$\mathbf{B} = \frac{\mu_0 J_0 \rho}{2}\hat{\boldsymbol{\varphi}} = \nabla\times\mathbf{A}$$
$$\mathbf{A} = -\frac{\mu_0 J_0\left(\rho^2 - a^2\right)}{4}\hat{\mathbf{z}}$$

Top view for $\rho > a$



$$\oint \mathbf{B}\cdot\mathbf{d}\ell = \mu_0 \int \mathbf{J}\cdot\mathbf{dA}$$
$$2\pi\rho B = \mu_0 J_0 \pi a^2$$
$$B = \frac{\mu_0 J_0 a^2}{2\rho}$$
$$\mathbf{B} = \frac{\mu_0 J_0 a^2}{2\rho}\hat{\boldsymbol{\varphi}} = \nabla\times\mathbf{A}$$
$$\mathbf{A} = -\frac{\mu_0 J_0 a^2 \ln\left(\rho/a\right)}{2}\hat{\mathbf{z}}$$

Comment on HW #11 -- continued



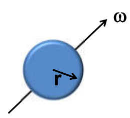
Alternative treatment using differential equations:

$$-\nabla^2\mathbf{A} = \begin{cases}\mu_0 J_0 \hat{\mathbf{z}} & \text{for } \rho \le a \\ 0 & \text{for } \rho > a\end{cases}$$
$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_z(\rho)}{\partial\rho} = \begin{cases}\mu_0 J_0 & \text{for } \rho \le a \\ 0 & \text{for } \rho > a\end{cases}$$
$$A_z(\rho) = \begin{cases}-\frac{\mu_0 J_0 \rho^2}{4} + C_1 & \text{for } \rho \le a \\ C_2 + C_3\ln(\rho) & \text{for } \rho > a\end{cases}$$

Choosing constants from continuity requirements:

$$A_z(\rho) = \begin{cases}-\frac{\mu_0 J_0 \rho^2}{4} + \frac{\mu_0 J_0 a^2}{4} & \text{for } \rho \le a \\ -\frac{\mu_0 J_0 a^2}{2}\ln(\rho/a) & \text{for } \rho > a\end{cases}$$
$$\mathbf{B} = -\frac{\partial A_z(\rho)}{\partial\rho}\hat{\boldsymbol{\varphi}}$$

Comment on HW #12



A sphere of radius a carries a uniform surface charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential **A** and magnetic field **B** both inside and outside the sphere.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \sigma \;\delta(r'-a)\; \boldsymbol{\omega}\times\mathbf{r}' & \text{for } r' \leq a \\ 0 & \text{otherwise} \end{cases}$$

Note that: $\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}')$

and: $\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}')\, \mathbf{r}' = \frac{r'}{r}\mathbf{r}\, \delta_{l1}.$

Comment on HW #12 -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0\sigma}{4\pi}\frac{\boldsymbol{\omega}\times\mathbf{r}}{r}\frac{4\pi}{3}\int_0^a r'^3 dr'\,\delta(r'-a)\frac{r_<}{r_>^2}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0\sigma}{3}\boldsymbol{\omega}\times\mathbf{r} \begin{cases} a & \text{for } r \leq a \\ \frac{a^4}{r^3} & \text{for } r > a \end{cases}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0\sigma}{3} \begin{cases} 2\boldsymbol{\omega}a & \text{for } r \leq a \\ \frac{a^4}{r^3}\left(3\left(\hat{\mathbf{r}}\cdot\boldsymbol{\omega}\right)\hat{\mathbf{r}}-\boldsymbol{\omega}\right) & \text{for } r > a \end{cases}$$