PHY 712 Electrodynamics
9-9:50 AM Olin 103

Plan for Lecture 17:

Read Chapter 7

1. Real and imaginary contributions to electromagnetic response
2. Frequency dependence of dielectric materials; Drude model
3. Kramers-Kronig relationships

9	Wed: 02/01/2017	Chap. 4	Dipoles and dielectrics	#9	02/03/2017
10	Fri: 02/03/2017	Chap. 4	Dipoles and dielectrics	#10	02/06/2017
11	Mon: 02/06/2017	Chap. 5	Magnetostatics	#11	02/08/2017
12	Wed: 02/08/2017	Chap. 5	Magnetostatics and the Hyperfine Interaction	#12	02/10/2017
13	Fri: 02/10/2017	Chap. 5	Magnetic dipoles and dipolar fields	#13	02/13/2017
14	Mon: 02/13/2017	Chap. 6	Maxwell's Equations	#14	02/15/2017
15	Wed: 02/15/2017	Chap. 6	Electromagnetic energy and forces	#15	02/17/2017
16	Fri: 02/17/2017	Chap. 7	Electromagnetic plane waves	#16	02/20/2017
17	Mon: 02/20/2017	Chap. 7	Dielectric media		
18	Wed: 02/22/2017	Chap. 7	Complex dielectrics		
19	Fri: 02/24/2017	Chap. 1-7	Review -- Take home exam distributed		
20	Mon: 02/27/2017				Exam
21	Wed: 03/01/2017				Exam
22	Fri: 03/03/2017				Exam Due
	Mon: 03/06/2017		*Spring break - no class*		
	Wed: 03/08/2017		*Spring break - no class*		
	Fri: 03/10/2017		*Spring break - no class*		
	Mon: 03/13/2017		*APS Meeting - no class*		
	Wed: 03/15/2017		*APS Meeting - no class*		
	Fri: 03/17/2017		*APS Meeting - no class*		
23	Mon: 03/20/2017				
24	Wed: 03/22/2017				

Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D}=\varepsilon\mathbf{E}; \quad \mathbf{B}=\mu\mathbf{H}$

Coulomb's law: $\nabla\cdot\mathbf{E}=0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

Plane wave solutions to sourceless Maxwell's equations; extension of analysis to complex dielectric functions

For simplicity assume that $\mu = \mu_0$

Suppose the dielectric function is complex:

$$\varepsilon = \varepsilon_R + i\varepsilon_I \qquad \frac{\varepsilon}{\varepsilon_0} = (n_R + i n_I)^2 \equiv \alpha + i\beta$$

$$n_R = \left(\frac{\sqrt{\alpha^2+\beta^2}+\alpha}{2}\right)^{1/2} \qquad n_I = \left(\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}\right)^{1/2}$$

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n_R\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) e^{-\frac{\omega}{c}n_I\hat{\mathbf{k}}\cdot\mathbf{r}}$$

Paul Karl Ludwig Drude 1863-1906

http://photos.aip.org/history/Thumbnails/drude_paul_a1.jpg

Drude model:

Vibrations of charged particles near equilibrium:

$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

For $\delta\mathbf{r} \equiv \delta\mathbf{r}_0 e^{-i\omega t}$, $\quad \delta\mathbf{r}_0 = \frac{q\mathbf{E}_0}{m}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}$

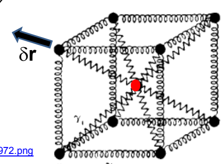
Induced dipole:

$$\mathbf{p} = q\,\delta\mathbf{r} = \frac{q^2\mathbf{E}_0}{m}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}e^{-i\omega t}$$

Displacement field:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$



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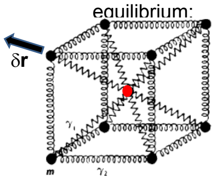
Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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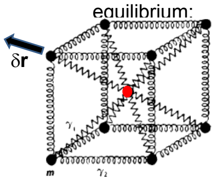
$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

Note that:

- $\gamma > 0$ represents dissipation of energy.
- ω_0 represents the natural frequency of the vibration; $\omega_0=0$ would represent a free (unbound) particle

Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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For $\delta\mathbf{r} \equiv \delta\mathbf{r}_0 e^{-i\omega t}$, $\quad \delta\mathbf{r}_0 = \frac{q\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

Induced dipole:

$$\mathbf{p} = q\,\delta\mathbf{r} = \frac{q^2\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}e^{-i\omega t}$$

Drude model:

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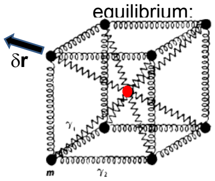

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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

Displacement field:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

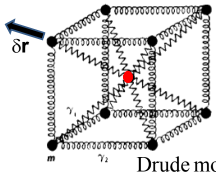
$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i) \approx N\sum_i f_i\,\mathbf{p}_i$$

$N \equiv$ number dipole/volume

$f_i \equiv$ fraction of type i dipoles

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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Drude model expression for permittivity :

$$\mathbf{D} = \varepsilon\mathbf{E} \quad = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\mathbf{E} + N\sum_i f_i\,\mathbf{p}_i$$

$$\mathbf{p}_i \;= q_i\;\delta\mathbf{r} = \frac{q_i^{\,2}\mathbf{E}_0}{m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}e^{-i\omega t}$$

$$\varepsilon\mathbf{E} \;\;= \varepsilon_0\mathbf{E}_0 e^{-i\omega t}\left(1+N\sum_i f_i\frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}\right)$$

Drude model dielectric function:

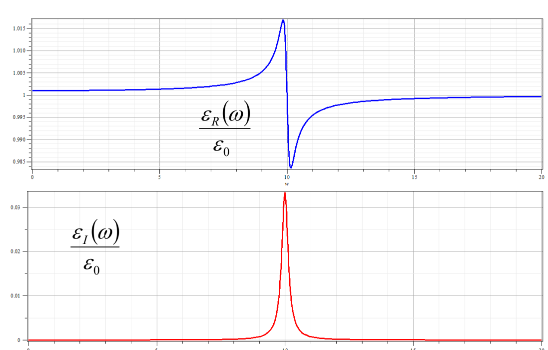
$$\frac{\varepsilon(\omega)}{\varepsilon_0} \;= 1+N\sum_i f_i\frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$$

$$=\frac{\varepsilon_R(\omega)}{\varepsilon_0}+i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0}=1+N\sum_i f_i\frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{\omega_i^2-\omega^2}{\left(\omega_i^2-\omega^2\right)^2+\omega^2\gamma_i^{\,2}}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0}=N\sum_i f_i\frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{\omega\gamma_i}{\left(\omega_i^2-\omega^2\right)^2+\omega^2\gamma_i^{\,2}}$$

Drude model dielectric function:



Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$\text{For } \omega >> \omega_i \quad \frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\right)$$

$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega_0 = 0$ (representing a free particle of charge q_0, mass m_0

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_{i>0} f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} + iNf_0 \frac{q_0^2}{\varepsilon_0 m_0} \frac{1}{\omega(\gamma_0 - i\omega)}$$

$$\equiv \frac{\varepsilon_b(\omega)}{\varepsilon_0} + i\frac{\sigma(\omega)}{\varepsilon_0 \omega}$$

Some details:

$\mathbf{D} = \varepsilon_b \mathbf{E} \qquad \mathbf{J} = \sigma \mathbf{E}$

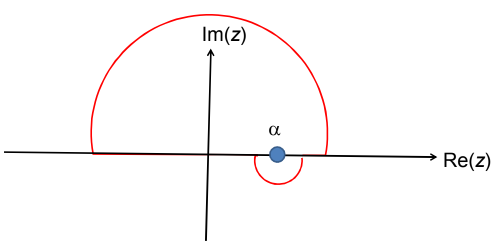
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = (\sigma - i\omega\varepsilon_b)\mathbf{E} = \varepsilon\frac{\partial \mathbf{E}}{\partial t} = -i\omega\left(\varepsilon_b + \frac{i\sigma}{\omega}\right)\mathbf{E}$$

$$\Rightarrow \sigma(\omega) = Nf_0 \frac{q_0^2}{m_0} \frac{1}{(\gamma_0 - i\omega)}$$

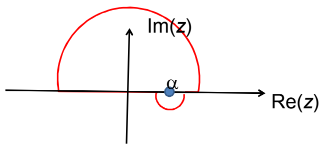
Analytic properties of the dielectric function (in the Drude model or from "first principles" -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function $f(z)$:

$$\oint dz\, f(z) = 0 \qquad f(\alpha) = \frac{1}{2\pi i} \oint_{includes\ \alpha} dz \frac{f(z)}{z-\alpha}$$



Kramers-Kronig transform -- continued



$$f(\alpha)=\frac{1}{2\pi i}\oint_{includes\,\alpha} dz\,\frac{f(z)}{z-\alpha} = \frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha} + \underbrace{\int_{rest} dz\,\frac{f(z)}{z-\alpha}}_{=0}\right)$$

$$f(\alpha)= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha} = \frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}+\frac{1}{2}f(\alpha)$$

Kramers-Kronig transform -- continued

$$f(\alpha)= \frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}+\frac{1}{2}f(\alpha)$$

Suppose $f(z_R)= f_R(z_R)+if_I(z_R)$:

$$\Rightarrow \frac{1}{2}\left(f_R(\alpha)+if_I(\alpha)\right)=\frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f_R(z_R)+if_I(z_R)}{z_R-\alpha}$$

$$\Rightarrow f_R(\alpha)=\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_I(z_R)}{z_R-\alpha}$$

$$f_I(\alpha)=-\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_R(z_R)}{z_R-\alpha}$$

Kramers-Kronig transform -- continued

$$f_R(\alpha)=\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_I(z_R)}{z_R-\alpha}$$

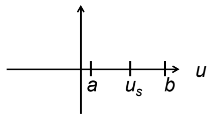
$$f_I(\alpha)=-\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_R(z_R)}{z_R-\alpha}$$

This Kramers-Kronig transform is useful for the dielectric function

when $f(z_R)\Rightarrow\dfrac{\varepsilon(\omega)}{\varepsilon_0}-1$

Must show that: 1. $f(z)$ is analytic for $z_I \geq 0$

2. $f(z)$ vanishes for $z\rightarrow\infty$



Some practical considerations

Principal parts integration :

$$P\int_a^b du\, g(u) = \lim_{\upsilon \to 0}\left(\int_a^{u_s-\upsilon} du\, g(u) + \int_{u_s+\upsilon}^{b} du\, g(u)\right)$$

Example :

$$P\int_a^b du\, \frac{1}{u\text{-}u_s} = \lim_{\upsilon \to 0}\left(\int_a^{u_s-\upsilon} du\, \frac{1}{u\text{-}u_s} + \int_{u_s+\upsilon}^{b} du\, \frac{1}{u\text{-}u_s}\right)$$

$$= \lim_{\upsilon \to 0}\left(\ln\left(\frac{\upsilon}{u_s-a}\right) + \ln\left(\frac{b-u_s}{\upsilon}\right)\right) = \ln\left(\frac{b-u_s}{u_s-a}\right)$$

More practical considerations

For dielectric function $\varepsilon(\omega)$:

$$\varepsilon(-\omega) = \varepsilon^*(\omega)$$
$$\Rightarrow \varepsilon_R(-\omega) = \varepsilon_R(\omega)$$
$$\Rightarrow \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Analytic properties the dielectric function which justify the treatment of $f(z) \Rightarrow \frac{\varepsilon(z)}{\varepsilon_0} - 1$

Must show that : 1. $f(z)$ is analytic for $z_I \geq 0$

2. $f(z)$ vanishes for $z \to \infty$ (for $z_I \geq 0$)

Analysis for Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let $f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For $|z| >> \omega_i$

$$f(z) \approx -\frac{1}{z^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\right) \Rightarrow \text{vanishes at large } z$$

Analysis for Drude model dielectric function – continued --
Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P{}^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P) \leq 0 \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1\right) \frac{1}{\omega' - \omega}$$

with $\varepsilon_R(-\omega) = \varepsilon_R(\omega)$; $\varepsilon_I(-\omega) = -\varepsilon_I(\omega)$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0\left\{\mathbf{E}(\mathbf{r},t) + \int_0^{\infty} d\tau\, G(\tau)\mathbf{E}(\mathbf{r},t-\tau)\right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^{\infty} d\tau\, G(\tau) e^{i\omega\tau}$$