

**PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103**

Plan for Lecture 14:

Start reading Chapter 6

- 1. Maxwell's full equations; effects of time varying fields and sources
- 2. Gauge choices and transformations
- 3. Green's function for vector and scalar potentials

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10	Fri: 02/03/2017	Chap. 4	Dipoles and dielectrics	#10	02/06/2017
11	Mon: 02/06/2017	Chap. 5	Magnetostatics	#11	02/08/2017
12	Wed: 02/08/2017	Chap. 5	Magnetostatics and the Hyperfine Interaction	#12	02/10/2017
13	Fri: 02/10/2017	Chap. 5	Magnetic dipoles and dipolar fields	#13	02/13/2017
14	Mon: 02/13/2017	Chap. 6	Maxwell's Equations	#14	02/15/2017
15	Wed: 02/15/2017				
16	Fri: 02/17/2017				
17	Mon: 02/20/2017				
18	Wed: 02/22/2017				
19	Fri: 02/24/2017				
20	Mon: 02/27/2017				Exam
21	Wed: 03/01/2017				Exam
22	Fri: 03/03/2017				Exam Due
	Mon: 03/06/2017	Spring break - no class			
	Wed: 03/08/2017	Spring break - no class			
	Fri: 03/10/2017	Spring break - no class			
	Mon: 03/13/2017	APS Meeting - no class			
	Wed: 03/15/2017	APS Meeting - no class			
	Fri: 03/17/2017	APS Meeting - no class			
23	Mon: 03/20/2017				

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Full electrodynamics with time varying fields and sources

Maxwell's equations



Image of statue of James Clerk-Maxwell in Edinburgh

"From a long view of the history of mankind - seen from, say, ten thousand years from now - there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics"

Richard P Feynman

<http://www.clerkmaxwellfoundation.org/>

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Maxwell's equations

Coulomb's law : $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere - Maxwell's law : $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

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Maxwell's equations

Microscopic or vacuum form ($\mathbf{P} = 0$; $\mathbf{M} = 0$):

Coulomb's law : $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$

Ampere - Maxwell's law : $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law : $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\epsilon_0 \mu_0}$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

or $\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\nabla \cdot \mathbf{E} = \rho / \epsilon_0 :$$

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \epsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

General form for the scalar and vector potential equations:

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \epsilon_0$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Coulomb gauge form -- require $\nabla \cdot \mathbf{A}_C = 0$

$$-\nabla^2 \Phi_C = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_C}{\partial t^2} + \frac{1}{c^2} \frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}$$

Note that $\mathbf{J} = \mathbf{J}_i + \mathbf{J}_t$ with $\nabla \times \mathbf{J}_i = 0$ and $\nabla \cdot \mathbf{J}_t = 0$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Coulomb gauge form -- require $\nabla \cdot \mathbf{A}_C = 0$

$$-\nabla^2 \Phi_C = \rho / \epsilon_0$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_C}{\partial t^2} + \frac{1}{c^2} \frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}$$

Note that $\mathbf{J} = \mathbf{J}_i + \mathbf{J}_t$ with $\nabla \times \mathbf{J}_i = 0$ and $\nabla \cdot \mathbf{J}_t = 0$

Continuity equation for charge and current density :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}_i = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}_t = -\epsilon_0 \nabla \cdot \frac{\partial(\nabla \Phi_C)}{\partial t}$$

$$\Rightarrow \frac{1}{c^2} \frac{\partial(\nabla \Phi_C)}{\partial t} = \epsilon_0 \mu_0 \frac{\partial(\nabla \Phi_C)}{\partial t} = \mu_0 \mathbf{J}_t$$

$$-\nabla^2 \mathbf{A}_C + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_C}{\partial t^2} = \mu_0 \mathbf{J}_t$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Review of the general equations:

$$-\nabla^2\Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \epsilon_0$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \left(\frac{\partial(\nabla\Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \mu_0 \mathbf{J}$$

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2} \frac{\partial^2\Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

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Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2} \frac{\partial^2\Phi_L}{\partial t^2} = \rho / \epsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

Alternate potentials: $\mathbf{A}'_L = \mathbf{A}_L + \nabla\Lambda$ and $\Phi'_L = \Phi_L - \frac{\partial\Lambda}{\partial t}$

Yields same physics provided that: $\nabla^2\Lambda - \frac{1}{c^2} \frac{\partial^2\Lambda}{\partial t^2} = 0$

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Solution of Maxwell's equations in the Lorentz gauge

$$\nabla^2\Phi_L - \frac{1}{c^2} \frac{\partial^2\Phi_L}{\partial t^2} = -\rho / \epsilon_0$$

$$\nabla^2\mathbf{A}_L - \frac{1}{c^2} \frac{\partial^2\mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{J}$$

Consider the general form of the 3-dimensional wave equation:

$$\nabla^2\Psi - \frac{1}{c^2} \frac{\partial^2\Psi}{\partial t^2} = -4\pi f$$

$\Psi(\mathbf{r}, t) \Rightarrow$ wave field $f(\mathbf{r}, t) \Rightarrow$ source

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Let Ψ represent Φ, A_x, A_y, A_z Let f represent ρ, J_x, J_y, J_z

$$\nabla^2 \Psi(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2 \Psi(\mathbf{r}, t)}{\partial t^2} = -4\pi f(\mathbf{r}, t)$$

Green's function :

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, t; \mathbf{r}', t') = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

Formal solution for field $\Psi(\mathbf{r}, t)$:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) + \int d^3 r' \int dt' G(\mathbf{r}, t; \mathbf{r}', t') f(\mathbf{r}', t')$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Determination of the form for the Green's function :

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, t; \mathbf{r}', t') = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

For the case of isotropic boundary values at infinity :

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'|\right)\right)$$

Formal solution for field $\Psi(\mathbf{r}, t)$:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) + \int d^3 r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'|\right)\right) f(\mathbf{r}', t')$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function:

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) G(\mathbf{r}, t; \mathbf{r}', t') = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \delta(t - t')$$

"Proof" -- Fourier analysis in the time domain -- note that

$$\delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')}$$

Define:

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} \tilde{G}(\mathbf{r}, \mathbf{r}', \omega)$$

$$\Rightarrow \left(\nabla^2 + \frac{\omega^2}{c^2} \right) \tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = -4\pi \delta^3(\mathbf{r} - \mathbf{r}')$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued) :

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

For the case of isotropic boundary values at infinity :

$$\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = \tilde{G}(\mathbf{r} - \mathbf{r}', \omega)$$

Further assuming that $\tilde{G}(\mathbf{r} - \mathbf{r}', \omega)$ is isotropic in $|\mathbf{r} - \mathbf{r}'| \equiv R$:

$$\left(\frac{1}{R} \frac{d^2}{dR^2} R + \frac{\omega^2}{c^2}\right)\tilde{G}(R, \omega) = -4\pi\delta^3(\mathbf{r} - \mathbf{r}')$$

Solution :
$$\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = \frac{1}{R} e^{\pm i\omega R/c}$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued) :

$$\tilde{G}(\mathbf{r}, \mathbf{r}', \omega) = \frac{1}{|\mathbf{r} - \mathbf{r}'|} e^{\pm i\omega|\mathbf{r} - \mathbf{r}'|/c}$$

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} \tilde{G}(\mathbf{r}, \mathbf{r}', \omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t')} \frac{1}{|\mathbf{r} - \mathbf{r}'|} e^{\pm i\omega|\mathbf{r} - \mathbf{r}'|/c}$$

$$= \frac{1}{|\mathbf{r} - \mathbf{r}'|} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t-t' \pm |\mathbf{r} - \mathbf{r}'|/c)} \right)$$

$$= \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta(t - t' \pm |\mathbf{r} - \mathbf{r}'|/c) = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - t \mp |\mathbf{r} - \mathbf{r}'|/c)$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r}, t; \mathbf{r}', t') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t \pm \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right)$$

Solution for field $\Psi(\mathbf{r}, t)$:

$$\Psi(\mathbf{r}, t) = \Psi_{f=0}(\mathbf{r}, t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(t' - \left(t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)\right) f(\mathbf{r}', t')$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Liénard-Wiechert potentials and fields --
 Determination of the scalar and vector potentials for a moving point particle (also see Landau and Lifshitz *The Classical Theory of Fields*, Chapter 8.)

Consider the fields produced by the following source: a point charge q moving on a trajectory $\mathbf{R}_q(t)$.

Charge density: $\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t))$

Current density: $\mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t))$, where $\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}$.



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Solution of Maxwell's equations in the Lorentz gauge -- continued

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \int d^3r' dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c))$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int \int d^3r' dt' \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c)).$$

We performing the integrations over first d^3r' and then dt' making use of the fact that for any function of t' ,

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t_r)|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

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Solution of Maxwell's equations in the Lorentz gauge -- continued

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}},$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$ $t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$

$$\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r),$$

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Comment on Lienard-Wiechert potential results

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

Note that for any function $F(x)$:

$$\int_{-\infty}^{\infty} F(x) \delta(x - x_0) dx = F(x_0)$$

Now consider a function $p(x)$, for which $p(x_i) = 0$ for $i = 1, 2, \dots$

$$\int_{-\infty}^{\infty} F(x) \delta(p(x)) dx = \int_{-\infty}^{\infty} F(x) \left(\sum_i \delta\left(x - x_i \frac{dp}{dx}\bigg|_{x_i}\right) \right) dx$$

$$= \sum_i \frac{F(x_i)}{\left| \frac{dp}{dx}\bigg|_{x_i} \right|}$$

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Comment on Lienard-Wiechert potential results -- continued

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$$

In this case we have: $\int_{-\infty}^{\infty} f(t') \delta(p(t')) dt' = \frac{f(t_r)}{\left| 1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|} \right|}$

where: $p(t') \equiv t' - \left(t - \frac{|\mathbf{r} - \mathbf{R}_q(t')|}{c} \right)$

$$\frac{dp(t')}{dt'} = 1 - \frac{\frac{d\mathbf{R}_q(t')}{dt'} \cdot (\mathbf{r} - \mathbf{R}_q(t'))}{c|\mathbf{r} - \mathbf{R}_q(t')|} \equiv 1 - \frac{\dot{\mathbf{R}}_q(t') \cdot (\mathbf{r} - \mathbf{R}_q(t'))}{c|\mathbf{r} - \mathbf{R}_q(t')|}$$

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Summary of results for fields due to moving charge –
Liénard Wiechert potentials

Resulting scalar and vector potentials:

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}$$

Notation: $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$
 $\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r)$
 $t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}$

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