**Electrodynamics – PHY712**

**Lecture 12 – magnetostatic examples**

**Reference: Chap. 5 in J. D. Jackson's textbook.**

## Calculation of the vector potential for a confined current density

If the current density $\mathbf{J}(\mathbf{r})$ is confined in space, the vector potential in the Coulomb gauge can be calculated from

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{1}$$

## Simple example of current density from a rotating charged sphere

Consider the following example corresponding to a rotating charged sphere of radius $a$, with $\rho_0$ denoting the uniform charge density within the sphere and $\boldsymbol{\omega}$ denoting the angular rotation of the sphere:

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \rho_0 \boldsymbol{\omega} \times \mathbf{r}' & \text{for } r' \leq a \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

In order to evaluate the vector potential (1) for this problem, we can make use of the expansion:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}'). \tag{3}$$

Noting that

$$\mathbf{r}' = r' \sqrt{\frac{4\pi}{3}} \left( Y_{1-1}(\hat{\mathbf{r}}') \frac{\hat{\mathbf{x}} + \mathbf{i}\hat{\mathbf{y}}}{\sqrt{2}} + Y_{11}(\hat{\mathbf{r}}') \frac{-\hat{\mathbf{x}} + \mathbf{i}\hat{\mathbf{y}}}{\sqrt{2}} + Y_{10}(\hat{\mathbf{r}}')\hat{\mathbf{z}} \right), \tag{4}$$

we see that the angular integral in Eq. (1) can be simplified with the use of the identity:

$$\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}')\, \mathbf{r}' = \frac{r'}{r} \mathbf{r}\, \delta_{l1}. \tag{5}$$

## Simple example of current density from a rotating charged sphere – continued

Therefore the vector potential for this system is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \rho_0 \boldsymbol{\omega} \times \mathbf{r}}{3r} \int_0^a dr' \, r'^3 \frac{r_<}{r_>^2}, \tag{6}$$

which can be evaluated as:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \left( \frac{a^2}{2} - \frac{3r^2}{10} \right) & \text{for } r \leq a \\ \\ \frac{\mu_0 \rho_0}{3} \boldsymbol{\omega} \times \mathbf{r} \frac{a^5}{5r^3} & \text{for } r \geq a \end{cases} . \tag{7}$$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \begin{cases} \frac{\mu_0 \rho_0}{3} \left[ \boldsymbol{\omega} \left( a^2 - \frac{6}{5} r^2 \right) + \frac{3}{5} \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \leq a \\ \frac{\mu_0 \rho_0}{3} \left[ -\boldsymbol{\omega} \frac{a^5}{5r^3} + \frac{3a^5}{5r^5} \mathbf{r}(\boldsymbol{\omega} \cdot \mathbf{r}) \right] & \text{for } r \geq a \end{cases} . \tag{8}$$

### Another example – current associated with an electron in a spherical atom

In this case, we assume that the current density is due to an electron in a bound atomic state with quantum numbers $|nlm_l\rangle$, as described by a wavefunction $\psi_{nlm_l}(\mathbf{r})$, where the azimuthal quantum number $m_l$ is associated with a factor of the form $\mathrm{e}^{im_l\phi}$. For such a wavefunction the quantum mechanical current density operator can be evaluated:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar}{2m_e i}\left(\psi^*_{nlm_l}\nabla\psi_{nlm_l} - \psi_{nlm_l}\nabla\psi^*_{nlm_l}\right). \tag{9}$$

Since the only complex part of this wavefunction is associated with the azimuthal quantum number, this can be written:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar}{2m_e ir\sin\theta}\left(\psi^*_{nlm_l}\frac{\partial}{\partial\phi}\psi_{nlm_l} - \psi_{nlm_l}\frac{\partial}{\partial\phi}\psi^*_{nlm_l}\right)\hat{\phi} = \frac{-e\hbar m_l\hat{\phi}}{m_e r\sin\theta}\left|\psi_{nlm_l}\right|^2. \tag{10}$$

where $m_e$ denotes the electron mass and $e$ denotes the magnitude of the electron charge.

## Current associated with an electron in a spherical atom – continued

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar m_l \hat{\phi}}{m_e r \sin\theta} \left|\psi_{nlm_l}(\mathbf{r})\right|^2 = \frac{-e\hbar m_l \hat{\mathbf{z}} \times \mathbf{r}}{m_e r^2 \sin^2\theta} \left|\psi_{nlm_l}(\mathbf{r})\right|^2 . \tag{11}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{-e\hbar m_l}{m_e}\right) \int d^3r' \, \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2 \sin^2\theta'} . \tag{12}$$

Note that for some atomic wavefunctions, $\psi_{nlm_l}(\mathbf{r}')$, the evaluation of the vector potential $\mathbf{A}(\mathbf{r}$ simplifies.

## Current associated with an electron in a spherical atom – continued

For example, consider the $|nlm = 211\rangle$ state of a H atom:

$$\psi_{211}(\mathbf{r}) = -\sqrt{\frac{1}{64\pi a^3}}\frac{r}{a}\mathrm{e}^{-r/(2a)}\sin\theta \mathrm{e}^{i\phi}, \tag{13}$$

and

$$\mathbf{J}(\mathbf{r}') = \frac{-e\hbar}{64 m_e \pi a^5}\mathrm{e}^{-r'/a}\ \hat{\mathbf{z}} \times \mathbf{r}', \tag{14}$$

where $a$ here denotes the Bohr radius. Using arguments similar to those above, we find that

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0 \hat{\mathbf{z}} \times \mathbf{r}}{192 m_e \pi a^5 r}\int_0^\infty dr'\ r'^3\ \mathrm{e}^{-r'/a}\ \frac{r_<}{r_>^2}. \tag{15}$$

This expression can be integrated to give:

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0 \hat{\mathbf{z}} \times \mathbf{r}}{8 m_e \pi r^3}\left[1 - \mathrm{e}^{-r/a}\left(1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{8a^3}\right)\right]. \tag{16}$$

## Current associated with an electron in a spherical atom – continued

Previous result:

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0 \hat{\mathbf{z}} \times \mathbf{r}}{8m_e \pi r^3} \left[ 1 - \mathrm{e}^{-r/a} \left( 1 + \frac{r}{a} + \frac{r^2}{2a^2} + \frac{r^3}{8a^3} \right) \right]. \tag{17}$$

Note that for $r \to \infty$ :

$$\mathbf{A}(\mathbf{r}) = \frac{-e\hbar\mu_0 \hat{\mathbf{z}} \times \mathbf{r}}{8m_e \pi r^3} = \frac{\mu_0}{4\pi} \left( -\frac{e\hbar}{2m_e} \right) \frac{\hat{\mathbf{z}} \times \mathbf{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}, \tag{18}$$

where

$$\mathbf{m} = \left( -\frac{e\hbar}{2m_e} \right) \hat{\mathbf{z}}. \tag{19}$$

More generally:

$$\mathbf{m} = \frac{1}{2} \int d^3r' \;\; \mathbf{r}' \times \mathbf{J}(\mathbf{r}'). \tag{20}$$

## Current associated with an electron in a spherical atom – continued

Note that the general form of the current density for a spherical atom is given by:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar m_l \hat{\phi}}{m_e r \sin\theta} \left|\psi_{nlm_l}(\mathbf{r})\right|^2 = \frac{-e\hbar m_l}{m_e} \frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{r \sin^2\theta} \left|\psi_{nlm_l}(\mathbf{r})\right|^2 . \tag{21}$$

In this case, the general form of the magnetic dipole moment is given by

$$\mathbf{m} = \frac{1}{2} \int d^3r' \;\; \mathbf{r}' \times \mathbf{J}(\mathbf{r}') = -\frac{e\hbar m_l}{2m_e} \hat{\mathbf{z}} \int d^3r' \left|\psi_{nlm_l}(\mathbf{r}')\right|^2 = -\frac{e\hbar}{2m_e} m_l \hat{\mathbf{z}}. \tag{22}$$

**Systematic multipole analysis of vector potential for a general confined current density $\mathbf{J}(\mathbf{r})$ (assuming $\nabla \cdot \mathbf{J}(\mathbf{r}) = 0$.**

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{23}$$

For field point $\mathbf{r}$ outside of extent of current density:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} \cdots . \tag{24}$$

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi} \left( \frac{1}{r} \int d^3r' \mathbf{J}(\mathbf{r}') + \frac{\mathbf{r}}{r^3} \cdot \int d^3r' \mathbf{r}'\mathbf{J}(\mathbf{r}') \ldots . \right) \tag{25}$$

Note that

$$\int d^3r' \mathbf{J}(\mathbf{r}') = 0 \tag{26}$$

$$\mathbf{r} \cdot \int d^3r' \mathbf{r}'\mathbf{J}(\mathbf{r}') = -\frac{1}{2}\mathbf{r} \times \int d^3r' \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \equiv \mathbf{m} \times \mathbf{r}. \tag{27}$$

## Magnetic dipolar field

The magnetic dipole moment is defined by

$$\mathbf{m} = \frac{1}{2}\int d^3r'\mathbf{r}' \times \mathbf{J}(\mathbf{r}'), \tag{28}$$

with the corresponding potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}, \tag{29}$$

and magnetostatic field

$$\mathbf{B}_{\mathbf{m}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}}) - \mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\delta^3(\mathbf{r})\right\}. \tag{30}$$

## Magnetic dipolar field – continued

Some details:

$$\nabla \times (s\mathbf{V}) = \nabla s \times \mathbf{V} + s\nabla \times \mathbf{V}. \tag{31}$$

$$\nabla \times (\mathbf{V}_1 \times \mathbf{V}_2) = \mathbf{V}_1(\nabla \cdot \mathbf{V}_2) - \mathbf{V}_2(\nabla \cdot \mathbf{V}_1) + (\mathbf{V}_2 \cdot \nabla)\mathbf{V}_1 - (\mathbf{V}_1 \cdot \nabla)\mathbf{V}_2. \tag{32}$$

For $r > 0$:

$$\nabla \times \left(\frac{\mathbf{m} \times \mathbf{r}}{r^3}\right) = \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r}) - r^2\mathbf{m}}{r^5}. \tag{33}$$

## Justification for the $\delta$ function contribution at the origin of the magnetic dipole

Note: This derivation is very similar to the analogous electrostatic case.

The evaluation of the field at the origin of the dipole is poorly defined, but we make the following approximation.

$$\mathbf{B}(\mathbf{r} \approx \mathbf{0}) \approx \left( \int_{\rm sphere} \mathbf{B}(\mathbf{r}) \mathbf{d^3 r} \right) \delta^{\mathbf{3}}(\mathbf{r}). \tag{34}$$

First we note that

$$\int_{r \leq R} \mathbf{B}(\mathbf{r}) d^3 r = R^2 \int_{r=R} \hat{\mathbf{r}} \times \mathbf{A}(\mathbf{r}) \, d\Omega. \tag{35}$$

This result follows from the divergence theorm:

$$\int_{\rm vol} \nabla \cdot \mathcal{V} \mathbf{d^3 r} = \int_{\rm surface} \mathcal{V} \cdot \mathbf{dA}. \tag{36}$$

## Singular contribution to dipolar field – continued

The divergence theorem can be used to prove Eq. (35) for each cartesian coordinate of $\nabla \times \mathbf{A}$ since $\nabla \times \mathbf{A} = \hat{\mathbf{x}}\,(\hat{\mathbf{x}} \cdot (\nabla \times \mathbf{A})) + \hat{\mathbf{y}}\,(\hat{\mathbf{y}} \cdot (\nabla \times \mathbf{A})) + \hat{\mathbf{z}}\,(\hat{\mathbf{z}} \cdot (\nabla \times \mathbf{A}))$. Note that $\hat{\mathbf{x}} \cdot (\nabla \times \mathbf{A}) = -\nabla \cdot (\hat{\mathbf{x}} \times \mathbf{A})$ and that we can use the Divergence theorem with $\mathcal{V} \equiv \hat{\mathbf{x}} \times \mathbf{A}(\mathbf{r})$ for the $x-$ component for example:

$$\int_{\text{vol}} \nabla \cdot (\hat{\mathbf{x}} \times \mathbf{A}) d^3r = \int_{\text{surface}} (\hat{\mathbf{x}} \times \mathbf{A}) \cdot \hat{\mathbf{r}} dA = \int_{\text{surface}} (\mathbf{A} \times \hat{\mathbf{r}}) \cdot \hat{\mathbf{x}} dA. \tag{37}$$

Therefore,

$$\int_{r \leq R} (\nabla \times \mathbf{A}) d^3r = -\int_{r=R} (\mathbf{A} \times \hat{\mathbf{r}}) \cdot (\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}) dA = R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega \tag{38}$$

which is identical to Eq. (35). We can use the identity (as in electrostatic case),

$$\int d\Omega \frac{\hat{\mathbf{r}}}{|\mathbf{r} - \mathbf{r}'|} = \frac{4\pi}{3} \frac{r_<}{r_>^2} \hat{\mathbf{r}}'. \tag{39}$$

## Singular contribution to dipolar field – continued

Now, expressing the vector potential in terms of the current density:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{40}$$

the integral over $\Omega$ in Eq. 35 becomes

$$R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega = \frac{4\pi R^2}{3} \frac{\mu_0}{4\pi} \int d^3r' \, \frac{r_<}{r_>^2} \, \hat{\mathbf{r}'} \times \mathbf{J}(\mathbf{r}'). \tag{41}$$

If the sphere $R$ contains the entire current distribution, then $r_> = R$ and $r_< = r'$ so that (41) becomes

$$R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega = \frac{4\pi}{3} \frac{\mu_0}{4\pi} \int d^3r' \, \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \equiv \frac{8\pi}{3} \frac{\mu_0}{4\pi} \mathbf{m}, \tag{42}$$

which thus justifies the delta-function contribution in Eq. 30 and results so-called "Fermi contact" contribution in the "hyperfine" interaction.

## Magnetic field due to electrons in the vicinity of a nucleus

## Contribution due to "orbital" magnetism in a spherical atom

The current density associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction $\psi_{nlm_l}(\mathbf{r})$ can be written:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar m_l \hat{\phi}}{m_e r \sin\theta} \left|\psi_{nlm_l}(\mathbf{r})\right|^2 . \quad (43)$$

In the following, it will be convenient to represent the azimuthal unit vector $\hat{\phi}$ in terms of cartesian coordinates:

$$\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} = \frac{\hat{\mathbf{z}} \times \mathbf{r}}{r\sin\theta}. \quad (44)$$

The vector potential for this current density can be written

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2 \sin^2\theta'} \quad (45)$$

## Contribution due to "orbital" magnetism in a spherical atom – continued

We want to evaluate the magnetic field $B = \nabla \times A$ in the vicinity of the nucleus $(\mathbf{r} \to 0)$. Taking the curl of the Eq. 45, we obtain

$$\mathbf{B_o}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar}{m_e}\, m_l \int d^3r' \frac{(\mathbf{r}-\mathbf{r}') \times (\hat{\mathbf{z}} \times \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{\mathbf{3}}} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'} \tag{46}$$

Evaluating this expression with $(\mathbf{r} \to 0)$, we obtain

$$\mathbf{B_o}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e}\, m_l \int d^3r' \frac{\mathbf{r}' \times (\hat{\mathbf{z}} \times \mathbf{r}')}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'} \tag{47}$$

## Contribution due to "orbital" magnetism in a spherical atom – continued

$$\mathbf{B_o}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e}\, m_l \int d^3r' \frac{\mathbf{r}' \times (\hat{\mathbf{z}} \times \mathbf{r}')}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'} \tag{48}$$

Expanding the cross product and expressing the result in spherical polar coordinates, we obtain in the numerator

$\hat{\mathbf{r}}' \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}}') = \hat{\mathbf{z}}(\mathbf{1} - \cos^{\mathbf{2}}\theta') - \hat{\mathbf{x}}\cos\theta' \sin\theta' \cos\phi' - \hat{\mathbf{y}}\cos\theta' \sin\theta' \sin\phi')$.

In evaluating the integration over the azimuthal variable $\phi'$, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components vanish which reduces to

$$\mathbf{B_o}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e}\, m_l \int d^3r' \frac{\hat{\mathbf{z}} r'^2 \sin^2\theta'}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'} \tag{49}$$

and

$$\mathbf{B_o}(\mathbf{0}) = -\frac{\mu_0 e\hbar m_l \hat{\mathbf{z}}}{4\pi m_e} \int d^3r' \, |\psi_{nlm_l}|^2 \frac{1}{r'^3} \equiv -\frac{\mu_0 e}{4\pi m_e} L_z \hat{\mathbf{z}} \left\langle \frac{1}{r'^3} \right\rangle. \tag{50}$$

## “Hyperfine” interaction

The so-called “hyperfine” interaction results from the magnetic dipole moment of a nucleus $\mu_{\mathbf{N}}$ responding to the magnetic field formed by the magnetic dipole of the electron spin ($\mu_{\mathbf{e}}$) as well as the electron orbital current contribution.

$$\mathcal{H}_{\mathrm{HF}} = -\mu_{\mathbf{N}} \cdot \left(\mathbf{B}_{\mu_e} + \mathbf{B}_o(0)\right). \tag{51}$$

$$\mathcal{H}_{\mathrm{HF}} = -\frac{\mu_0}{4\pi}\left(\frac{3(\mu_{\mathbf{N}}\cdot\hat{\mathbf{r}})(\mu_{\mathbf{e}}\cdot\hat{\mathbf{r}}) - \mu_{\mathbf{N}}\cdot\mu_{\mathbf{e}}}{r^3} + \frac{8\pi}{3}\mu_{\mathbf{N}}\cdot\mu_{\mathbf{e}}\delta^3(\mathbf{r}) + \frac{e}{m_e}\left\langle\frac{\mathbf{L}\cdot\mu_{\mathbf{N}}}{r^3}\right\rangle\right). \tag{52}$$