

PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 7:

Start reading Chapter 3

Solution of Poisson equation in for special geometries –

A. Cylindrical

B. Spherical

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PHY 712 Electrodynamics

MWF 9-9:50 AM OPL 103 | <http://www.wfu.edu/~natale/s17phy712/>

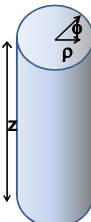
Instructor: [Natalie Holzwarth](mailto:natale@wfu.edu) Phone: 758-5510 / Office: 300 OPL e-mail: natale@wfu.edu

Course schedule for Spring 2017
 (Preliminary schedule – subject to frequent adjustment.)

Lecture date	JDJ Reading	Topic	HW	Due date
1 Wed: 01/11/2017	Chap. 1	Introduction, units and Poisson equation	#1	01/18/2017
2 Fri: 01/13/2017	Chap. 1	Electrostatic energy calculations	#2	01/18/2017
Mon: 01/16/2017		MLK Holiday - no class		
3 Wed: 01/18/2017	Chap. 1	Poisson equation and Green's theorem	#3	01/20/2017
4 Fri: 01/20/2017	Chap. 1 and 2	Poisson equation in 2 and 3 dimensions	#4	01/23/2017
5 Mon: 01/23/2017	Chap. 1 and 2	Brief introduction to grid solution methods	#5	01/25/2017
6 Wed: 01/25/2017	Chap. 2	Method of images	#6	01/27/2017
7 Fri: 01/27/2017	Chap. 3	Cylindrical and spherical geometries	#7	01/30/2017
8 Mon: 01/30/2017				

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Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with z-dependence



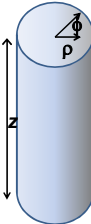
Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$$

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Cylindrical geometry continued:



Laplace equation : $\nabla^2\Phi = 0$
 $\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$
 One possibility :

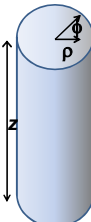
$$\frac{d^2Z}{dz^2} - k^2Z = 0 \quad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2} + m^2Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \quad \Rightarrow J_m(k\rho), N_m(k\rho)$$

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Cylindrical geometry continued:



Laplace equation : $\nabla^2\Phi = 0$
 $\Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$
 Another possibility :

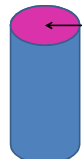
$$\frac{d^2Z}{dz^2} + k^2Z = 0 \quad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2Q}{d\phi^2} + m^2Q = 0 \quad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right)R = 0 \quad \Rightarrow I_m(k\rho), K_m(k\rho)$$

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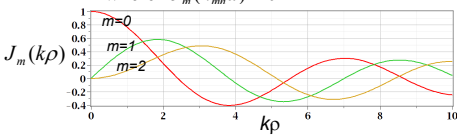
Solutions of Laplace equation inside cylindrical shape
 Example with non-trivial boundary value at $z=L$



$\Phi(\rho, \phi, z=L) = V(\rho, \phi)$
 $\Phi(\rho, \phi, z) = 0$ on all other boundaries


$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{nm} J_m(k_{nm}\rho) \sinh(k_{nm}z) \sin(m\phi + \alpha_{nm})$$

where $J_m(k_{nm}, \alpha) = 0$



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Solutions of Laplace equation inside cylindrical shape
 Example with non-trivial boundary value at $z=L$




$\Phi(\rho, \phi, z=L) = V(\rho, \phi)$
 $\Phi(\rho, \phi, z) = 0$ on all other boundaries
 $\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} J_m(k_{mn} \rho) \sinh(k_{mn} z) \sin(m\phi + \alpha_{mn})$
 If $V(\rho, \phi)$ is an even function of ϕ so that $\alpha_{mn} = \pi/2$:

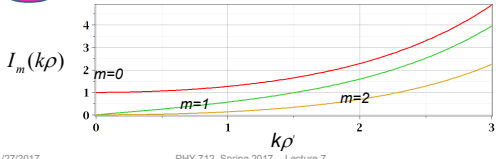
$$A_{mn} = \frac{\int_0^{2\pi} d\phi \cos(m\phi) \int_0^a \rho d\rho J_m(k_{mn} \rho) V(\rho, \phi)}{\sinh(k_{mn} L) \int_0^{2\pi} d\phi \cos^2(m\phi) \int_0^a \rho d\rho J_m^2(k_{mn} \rho)}$$

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Solutions of Laplace equation inside cylindrical shape
 Example with non-trivial boundary value at $\rho=a$

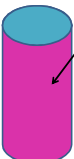


$\Phi(\rho = a, \phi, z) = V(\phi, z)$
 $\Phi(\rho, \phi, z) = 0$ on all other boundaries
 $\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \sin(m\phi + \alpha_{mn})$



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Solutions of Laplace equation inside cylindrical shape
 Example with non-trivial boundary value at $\rho=a$

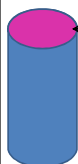


$\Phi(\rho = a, \phi, z) = V(\phi, z)$
 $\Phi(\rho, \phi, z) = 0$ on all other boundaries
 $\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right) \sin\left(\frac{n\pi z}{L}\right) \sin(m\phi + \alpha_{mn})$
 If $V(z, \phi)$ is an even function of ϕ so that $\alpha_{mn} = \pi/2$:

$$A_{mn} = \frac{\int_0^{2\pi} d\phi \cos(m\phi) \int_0^L dz \sin\left(\frac{n\pi z}{L}\right) V(z, \phi)}{I_m\left(\frac{n\pi a}{L}\right) \int_0^{2\pi} d\phi \cos^2(m\phi) \int_0^L dz \sin^2\left(\frac{n\pi z}{L}\right)}$$

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Green's function for Dirchelet boundary value inside cylindrar:



$$\Phi(\rho, \phi, z=L) = V(\rho, \phi)$$

$$\Phi(\rho = a, \phi, z) = 0, \Phi(\rho, \phi, z = 0) = 0$$

Expansion in terms of Bessel function zeros: $J_m(k_{mn}a) = 0$

$$G(\rho, \rho', \phi, \phi', z, z') =$$

$$\frac{8\pi}{\pi a^2} \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{e^{im(\phi-\phi')} J_m(k_{mn}\rho) J_m(k_{mn}\rho') \sinh(k_{mn}z) \sinh(k_{mn}(L-z))}{k_{mn} (J_{m+1}(k_{mn}a))^2 \sinh(k_{mn}L)}$$

$$\Phi(\rho, \phi, z) = \frac{1}{4\pi\epsilon_0} \int_V d\phi' \rho' d\rho' dz' G(\rho, \rho', \phi, \phi', z, z') \rho(\rho', \phi', z')$$

$$+ \frac{1}{4\pi} \int_{S, z=L} d\phi' \rho' d\rho' \frac{\partial G(\rho, \rho', \phi, \phi', z, z')}{\partial z'} \Big|_{z=L} V(\rho', \phi')$$

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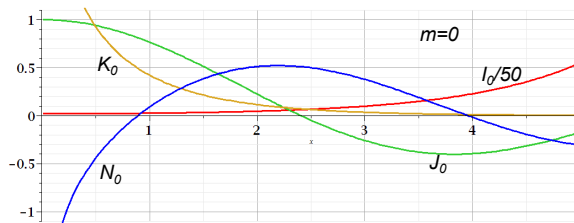
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Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2} \right) \right) F_m^{\pm}(u) = 0$$

$$F_m^+(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$$

$$F_m^-(u) = I_m(u), K_m(u)$$



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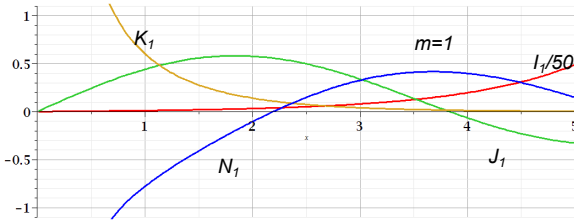
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Comments on cylindrical Bessel functions

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$$F_m^-(u) = I_m(u), K_m(u)$$



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Some useful identities involving cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u} \frac{d}{du} + \left(1 - \frac{m^2}{u^2}\right)\right) J_m(u) = 0 \quad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: x_{mn} ; $J_m(x_{mn}) = 0$

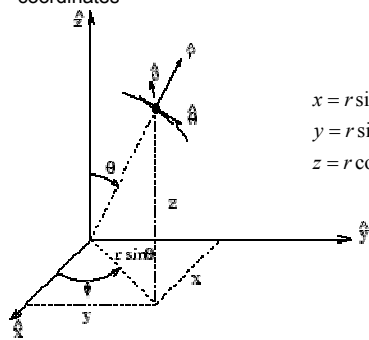
$$\int_0^a \rho d\rho J_m(x_{mn}\rho/a) J_m(x_{m'n'}\rho/a) = \frac{a^2}{2} (J_{m+1}(x_{mn}))^2 \delta_{nn'}$$

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Poisson and Laplace equation in spherical polar coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

<http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html>

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Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r, \theta, \phi)$:

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} (r\Phi) + \frac{1}{r^2} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \Phi = 0$$

$$\Phi(r, \theta, \phi) = \sum_{lm} R_{lm}(r) Y_{lm}(\theta, \phi)$$

Spherical harmonic functions:

$$\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y_{lm}(\theta, \phi) = -l(l+1) Y_{lm}(\theta, \phi)$$

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Properties of spherical harmonic functions

$Y_{lm}(\theta, \phi) = (-1)^m Y_{l(-m)}^*(\theta, \phi)$ (standard Condon - Shortley convention)

$\int d\Omega Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) \equiv \int \sin\theta d\theta d\phi Y_{lm}(\theta, \phi) Y_{l'm'}^*(\theta, \phi) = \delta_{ll'} \delta_{mm'}$

Completeness :

$\sum_{lm} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') \equiv \delta(\cos\theta - \cos\theta') \delta(\phi - \phi')$

Relationship to Legendre polynomials :

$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$

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Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r \rightarrow \infty$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') \right)$$

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Example -- continued

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') \right)$$

Suppose: $\rho(\mathbf{r}') = \frac{Q}{a^3 \pi^{3/2}} e^{-r'^2/a^2}$

$\int d\Omega Y_{lm}^*(\theta', \phi') = \sqrt{4\pi} \delta_{l0} \delta_{m0}$

$$\Phi(\mathbf{r}) = \frac{4\pi}{4\pi\epsilon_0} \int_0^\infty r'^2 dr' \int_{r_{>}^{-1}}^{r_{<}^0} \frac{Q}{a^3 \pi^{3/2}} e^{-r'^2/a^2}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{\text{erf}(r/a)}{r}$$

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Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Elements of "proof":

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \frac{1}{r_{>} \left(1 + \left(\frac{r_{<}}{r_{>}} \right)^2 - 2 \left(\frac{r_{<}}{r_{>}} \right) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' \right)^{1/2}} \\ &= \frac{1}{r_{>}} \left(1 + \left(\frac{r_{<}}{r_{>}} \right) \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' + \left(\frac{r_{<}}{r_{>}} \right)^2 \left(\frac{3}{2} (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 - \frac{1}{2} \right) + \dots \right) \\ &= \frac{1}{r_{>}} \left(\sum_{l=0}^{\infty} \left(\frac{r_{<}}{r_{>}} \right)^l P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') \right) \end{aligned}$$

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Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

Elements of "proof" -- continued:

Sum rule for spherical harmonics:

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}} = \hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') = 1$

$$\Rightarrow \frac{4\pi}{2l+1} \sum_{m=-l}^l |Y_{lm}(\hat{\mathbf{r}})|^2 = 1$$

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Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{2(\pm 2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$$

$$Y_{2(\pm 1)}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

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