PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 6:

Continue reading Chapter 2

1. **Methods of images -- planes, spheres**
2. **Solution of Poisson equation in for other geometries -- cylindrical**

PHY 712 Electrodynamics

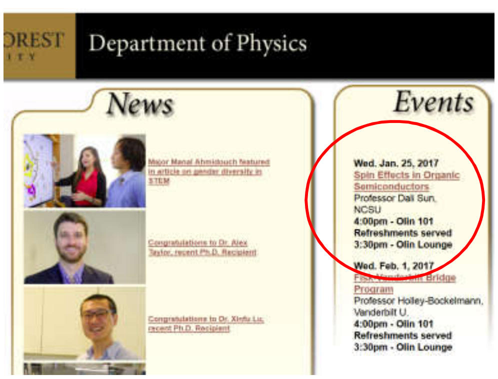
MWF 9-9:50 AM | OPL 103 | http://www.wfu.edu/~natalie/s17phy712/

Instructor: Natalie Holzwarth | Phone:758-5510 | Office:300 OPL | e-mail:natalie@wfu.edu

Course schedule for Spring 2017

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	HW	Due date
1	Wed: 01/11/2017	Chap. 1	Introduction, units and Poisson equation	#1	01/18/2017
2	Fri: 01/13/2017	Chap. 1	Electrostatic energy calculations	#2	01/18/2017
	Mon: 01/16/2017		*MLK Holiday - no class*		
3	Wed: 01/18/2017	Chap. 1	Poisson equation and Green's theorem	#3	01/20/2017
4	Fri: 01/20/2017	Chap. 1 and 2	Poisson equation in 2 and 3 dimensions	#4	01/23/2017
5	Mon: 01/23/2017	Chap. 1 and 2	Brief introduction to grid solution methods	#5	01/25/2017
6	Wed: 01/25/2017	Chap. 2	Method of images	#6	01/27/2017
7	Fri: 01/27/2017				
8	Mon: 01/30/2017				



Survey of mathematical techniques for analyzing electrostatics – the Poisson equation

$$\nabla^2\Phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

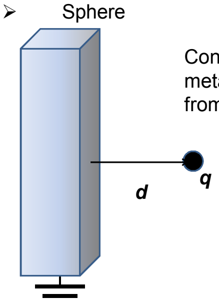
1. Direct integration of differential equation
2. Green's function techniques
3. Orthogonal function expansions
4. Numerical methods (finite differences and finite element methods)
5. Method of images

Method of images

Clever trick for specialized geometries:

- Flat plane (surface)
- Sphere

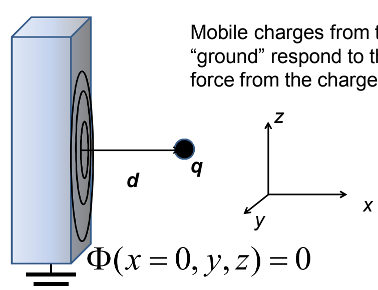
Planar case:



Consider a grounded metal sheet, a distance ***d*** from a point charge ***q***.

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

Mobile charges from the "ground" respond to the force from the charge ***q***.



$$\Phi(x=0,y,z)=0$$

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

$$\nabla^2\Phi = -\frac{q}{\varepsilon_0}\delta^3(\mathbf{r}-d\hat{\mathbf{x}})$$

$$\Phi(x=0,y,z)=0$$

Trick for $x \geq 0$:

$$\Phi(x\geq 0,y,z)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-d\hat{\mathbf{x}}|}-\frac{q}{|\mathbf{r}+d\hat{\mathbf{x}}|}\right)$$

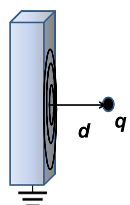
Surface charge density:

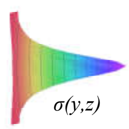
$$\sigma(y,z)=\varepsilon_0 E(0,y,z)=-\varepsilon_0\frac{d\Phi(0,y,z)}{dx}=-\frac{q}{4\pi}\left(\frac{2d}{(d^2+y^2+z^2)^{3/2}}\right)$$

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

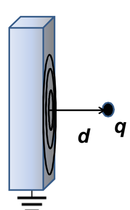
Surface charge density: $\sigma(y,z)=-\frac{q}{4\pi}\left(\frac{2d}{(d^2+y^2+z^2)^{3/2}}\right)$

Note: $\iint dydz\ \sigma(y,z)=-\frac{q2d}{4\pi}2\pi\int_0^\infty\frac{udu}{(d^2+u^2)^{3/2}}=-q$





A grounded metal sheet, a distance ***d*** from a point charge ***q***.



Surface charge density:

$$\sigma(y,z)=-\frac{q}{4\pi}\left(\frac{2d}{(d^2+y^2+z^2)^{3/2}}\right)$$

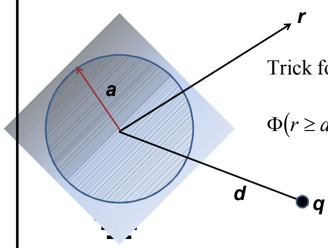
Force between charge and sheet:

$$\mathbf{F}=\frac{-q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0(2d)^2}$$

Image potential between charge and sheet at distance x:

$$V(x)=\frac{-q^2}{4\pi\varepsilon_0(4x)}$$

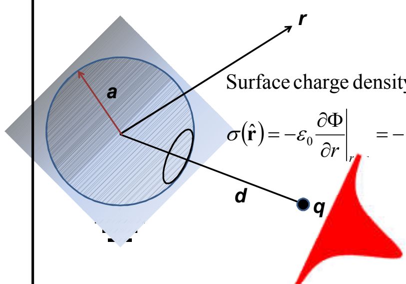
A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Trick for $r \geq a$:

$$\Phi(r \geq a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{\frac{d}{a}\left|\mathbf{r}-\mathbf{d}\frac{a^2}{d^2}\right|}\right)$$

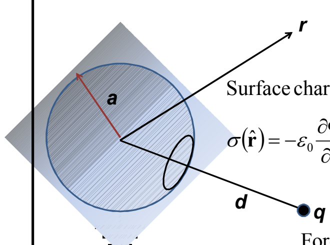
A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Surface charge density:

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Force between sphere and q:

$$|\mathbf{F}| = \frac{q^2}{4\pi\varepsilon_0}\frac{ad}{\left(d^2-a^2\right)^2}$$

Use of image charge formalism to construct Green's function

Example:

Suppose we have a Dirichlet boundary value problem on a sphere of radius a:

$$\nabla^2\Phi = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \Phi(r=a)=0$$

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow \text{For} \quad r,r' > a: \qquad G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\frac{r'}{a}\left|\mathbf{r} - \frac{a^2}{r'^2}\mathbf{r}'\right|}$$

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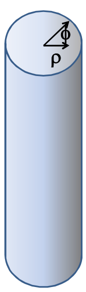
Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\Phi(\rho,\phi) = \Phi(\rho,\phi+m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation in these coordinates:

$$\Phi(\rho,\phi) = \mathrm{A}_0 + \mathrm{B}_0\ln(\rho) + \sum_{m=1}^{\infty}\left(A_m\rho^m + B_m\rho^{-m}\right)\sin(m\phi+\alpha_m)$$

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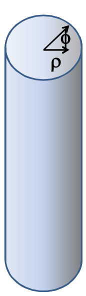
Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Green's function appropriate for this geometry with boundary conditions at $\rho = 0$ and $\rho = \infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G(\rho,\rho',\phi,\phi') = -4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\phi-\phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>^2\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos(m(\phi-\phi'))$$

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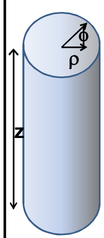
Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence



Corresponding orthogonal functions from solution of Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

$$\Phi(\rho,\phi,z)=\Phi(\rho,\phi+m2\pi,z)$$

$$\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$$

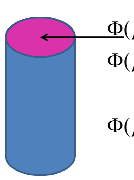
Cylindrical geometry continued:

$$\frac{d^2Z}{dz^2}-k^2Z=0 \qquad \Rightarrow Z(z)=\sinh(kz),\cosh(kz),e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2}+m^2Q=0 \qquad \Rightarrow Q(\phi)=e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(k^2-\frac{m^2}{\rho^2}\right)R=0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry example:

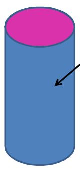


$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$\Phi(\rho,\phi,z)=0$ on all other boundaries

$$\Phi(\rho,\phi,z)=\sum_{n,m}A_{mn}J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin(m\phi+\alpha_{mn})$$

Cylindrical geometry example:



$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

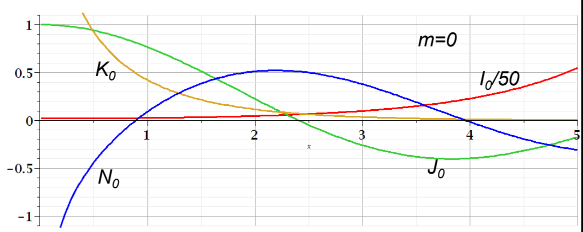
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin(m\phi + \alpha_{mn})$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2} + \frac{1}{u}\frac{d}{du} + \left(\pm 1 - \frac{m^2}{u^2}\right)\right)F_m^{\pm}(u) = 0$$

$$F_m^{+}(u) = J_m(u), N_m(u), H_m(u) \equiv J_m(u) \pm iN_m(u)$$

$$F_m^{-}(u) = I_m(u), K_m(u)$$



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