PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 36:

Review Part II:

- **Further comment of Kramers-Kronig transform**
- **Some equations for top of your head**
- **Example problems**
- **Course evaluation forms**

20	Mon: 03/16/2015	Chap. 8	Review Exam; Wave guides	#19	03/18/2015
21	Wed: 03/18/2015	Chap. 8	Wave guides	#20	03/20/2015
22	Fri: 03/20/2015	Chap. 9	Radiation sources	#21	03/23/2015
23	Mon: 03/23/2015	Chap. 9 & 10	Radiation and scattering	#22	03/25/2015
24	Wed: 03/25/2015	Chap. 9 & 10	Radiation and scattering		
25	Fri: 03/27/2015	Chap. 11	Special relativity	#23	03/30/2015
26	Mon: 03/30/2015	Chap. 11	Special relativity	#24	04/01/2015
27	Wed: 04/01/2015	Chap. 11	Special relativity	#25	04/06/2015
	Fri: 04/03/2015	Good Friday	No class		
28	Mon: 04/06/2015	Chap. 14	Radiation from moving charges	#26	04/08/2015
29	Wed: 04/08/2015	Chap. 14	Radiation from moving charges	#27	04/10/2015
30	Fri: 04/10/2015	Chap. 14	Radiation from moving charges	#28	04/13/2015
31	Mon: 04/13/2015	Chap. 15	Radiation due to scattering	#29	04/15/2015
32	Wed: 04/16/2015	Chap. 13	Cherenkov radiation	#30	04/17/2015
33	Fri: 04/17/2015		Special topics -- superconductivity		
34	Mon: 04/20/2015		Special topics -- superconductivity		
35	Wed: 04/22/2015		Review		
36	Fri: 04/24/2015		Review		
	Mon: 04/27/2015		Presentations I		
	Wed: 04/29/2015		Presentations II		
	Fri: 05/01/2015		Presentations III & Take home exam		

Schedule for PHY 712 Presentations

Monday 4/27/2015

	Presenter	Topic
9:00 -9:25 AM	Larry Rush	"Superconductivity"
9:25-9:50 AM	Junwei Xu	"Electrodynamics in alternating current electroluminescent device"

Wednesday 4/29/2015

	Presenter	Topic
9:00 -9:25 AM	Jason Howard	"Ewald summations with anisotropic dielectric screening"
9:25-9:50 AM	Eric Chapman	The Physics of MRI

Friday 5/1/2015

	Presenter	Topic
9:00 -9:25 AM	Lauren Nelson	Solar Cells
9:25-9:50 AM	Hysun Lee	Surface Plasmon and It's application

Review topic – analytic properties of dielectric function
Material from Chapter 7 in Jackson

The displacement field **D** is related to the electric field **E**

$$\mathbf{D} = \varepsilon\mathbf{E} \quad = \varepsilon_0\mathbf{E} + \mathbf{P}$$

Dielectric function $\varepsilon(\omega) = \varepsilon_R(\omega) + i\varepsilon_I(\omega)$

can be shown to be analyic for $\omega \to z$ for $\Im(z) > 0$

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1\right) \frac{1}{\omega' - \omega}$$

with $\varepsilon_R(-\omega) = \varepsilon_R(\omega)$; $\varepsilon_I(-\omega) = -\varepsilon_I(\omega)$

Practical evaluation of Kramers-Kronig relation

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1\right) \frac{1}{\omega' - \omega}$$

with $\varepsilon_R(-\omega) = \varepsilon_R(\omega)$; $\varepsilon_I(-\omega) = -\varepsilon_I(\omega)$

Let $\varepsilon_1(\omega) = \frac{\varepsilon_R(\omega)}{\varepsilon_0}$ $\quad \varepsilon_2(\omega) = \frac{\varepsilon_I(\omega)}{\varepsilon_0}$

$$\varepsilon_1(\omega) - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\varepsilon_2(\omega')}{\omega' - \omega} d\omega' = \frac{2}{\pi} P \int_0^{\infty} \frac{\omega'\varepsilon_2(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$\varepsilon_2(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\varepsilon_1(\omega') - 1}{\omega' - \omega} d\omega' = -\frac{2}{\pi} P \int_0^{\infty} \frac{\varepsilon_1(\omega') - 1}{\omega'^2 - \omega^2} d\omega'$$

Practical evaluation of Kramers-Kronig relation

$$\varepsilon_1(\omega) - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\varepsilon_2(\omega')}{\omega' - \omega} d\omega'$$

$$= \frac{1}{\pi} P \left(\int_0^{\infty} \frac{\varepsilon_2(\omega')}{\omega' - \omega} d\omega' + \int_{-\infty}^{0} \frac{\varepsilon_2(\omega')}{\omega' - \omega} d\omega' \right)$$

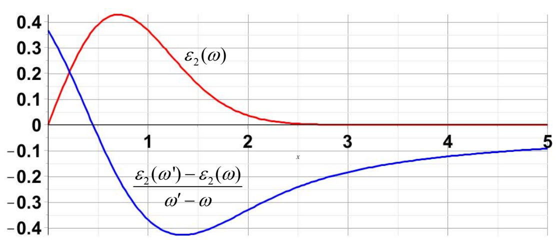
$$= \frac{1}{\pi} P \left(\int_0^{\infty} \frac{\varepsilon_2(\omega')}{\omega' - \omega} d\omega' + \int_0^{\infty} \frac{\varepsilon_2(\omega')}{\omega' + \omega} d\omega' \right)$$

Singular integral can be evaluated numerically:

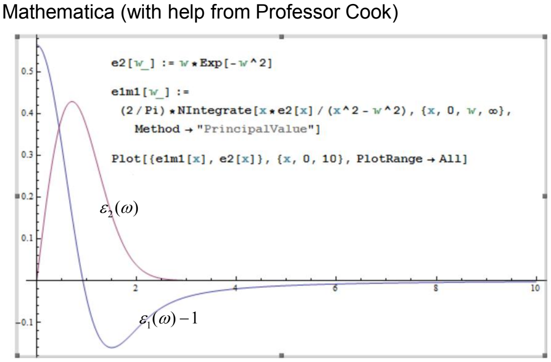
$$P \int_0^{\infty} \frac{\varepsilon_2(\omega')}{\omega' - \omega} d\omega' = P \int_0^{W} \frac{\varepsilon_2(\omega') - \varepsilon_2(\omega)}{\omega' - \omega} d\omega' + \varepsilon_2(\omega) \ln\left(\left| \frac{W - \omega}{\omega} \right| \right) + \int_W^{\infty} \frac{\varepsilon_2(\omega')}{\omega' - \omega} d\omega'$$

Evaluation of singular integral numerically:

$$P\int_0^\infty \frac{\varepsilon_2(\omega')}{\omega'-\omega}d\omega' = P\int_0^W \frac{\varepsilon_2(\omega')-\varepsilon_2(\omega)}{\omega'-\omega}d\omega' + \varepsilon_2(\omega)\ln\left(\left|\frac{W-\omega}{\omega}\right|\right) + \int_W^\infty \frac{\varepsilon_2(\omega')}{\omega'-\omega}d\omega'$$



Evaluation of Kramer's Kronig transform using Mathematica (with help from Professor Cook)

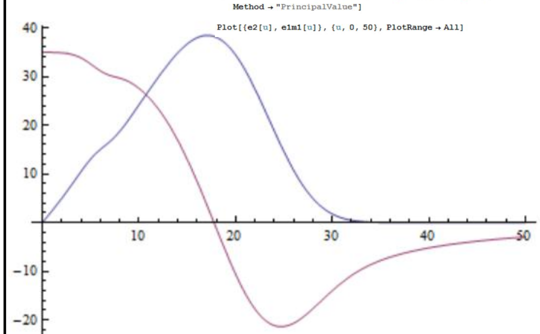


Another example

```
e2[x_] :=
 (x/4)*(2*Exp[-(((x/4)^2-1)^2)/3]+10*Exp[-(((x/13)^2-1)^2)/5])

e1m1[y_] :=
 (2/Pi)*NIntegrate[x*e2[x]/(x^2-y^2), {x, 0, y, ∞},
   Method → "PrincipalValue"]

Plot[{e2[u], e1m1[u]}, {u, 0, 50}, PlotRange → All]
```



Some equations worth remembering --

Maxwell's equations

SI units; Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

Maxwell's equations

SI units; Macroscopic form $(\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}=0;\ \mathbf{H}=\frac{1}{\mu_0}\mathbf{B}-\mathbf{M})$:

Coulomb's law: $\nabla\cdot\mathbf{D}=\rho_{free}$

Ampere-Maxwell's law: $\nabla\times\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}=\mu_0\mathbf{J}_{free}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

Maxwell's equations

Gaussian units; Macroscopic form ($\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = 0$; $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$):

Coulomb's law: $\nabla \cdot \mathbf{D} = 4\pi\rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

04/24/2015 PHY 712 Spring 2015 -- Lecture 36 13

Energy and power (SI units)

Electromagnetic energy density: $u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right)$

Poynting vector: $\mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$\left\langle u(\mathbf{r},t)\right\rangle_{t\text{ avg}} = \frac{1}{4}\Re\left(\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\cdot\tilde{\mathbf{D}}^*(\mathbf{r},\omega) + \tilde{\mathbf{B}}(\mathbf{r},\omega)\cdot\tilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$

$$\left\langle \mathbf{S}(\mathbf{r},t)\right\rangle_{t\text{ avg}} = \frac{1}{2}\Re\left(\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$

04/24/2015 PHY 712 Spring 2015 -- Lecture 36 14

Solution of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

Introduction of vector and scalar potentials:

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\Phi \quad \text{or} \quad \mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$$

04/24/2015 PHY 712 Spring 2015 -- Lecture 36 15

Scalar and vector potentials continued:

$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$:

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = \mu_0 \mathbf{J}$$

Analysis of the scalar and vector potential equations :

$$-\nabla^2 \Phi - \frac{\partial(\nabla \cdot \mathbf{A})}{\partial t} = \rho / \varepsilon_0$$

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}\left(\frac{\partial(\nabla \Phi)}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) = \mu_0 \mathbf{J}$$

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

Solution methods for scalar and vector potentials
and their electrostatic and magnetostatic analogs:

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

In your "bag" of tricks:

- ❑ **Direct (analytic or numerical) solution of differential equations**
- ❑ **Solution by expanding in appropriate orthogonal functions**
- ❑ **Green's function techniques**

How to choose most effective solution method --

- In general, Green's functions methods work well when source is contained in a finite region of space

Consider the electrostatic problem:

$$-\nabla^2\Phi_L = \rho / \varepsilon_0$$

Define: $\nabla'^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$

$$\Phi_L(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3r'\,\rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}') + \frac{1}{4\pi}\int_S d^2r'\left[G(\mathbf{r},\mathbf{r}')\nabla'\Phi(\mathbf{r}') - \Phi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]\cdot\hat{\mathbf{r}}'.$$

For electrostatic problems where $\rho(\mathbf{r})$ is contained in a small region of space and $S \to \infty$, $\quad G(\mathbf{r},\mathbf{r}') = \dfrac{1}{|\mathbf{r}-\mathbf{r}'|}$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Electromagnetic waves from time harmonic sources

Charge density : $\rho(\mathbf{r},t) = \Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Current density : $\mathbf{J}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Note that the continuity condition :

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{J}(\mathbf{r},t) = 0 \quad \Rightarrow -i\omega\tilde{\rho}(\mathbf{r},\omega) + \nabla\cdot\tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$$

For dynamic problems where $\tilde{\rho}(\mathbf{r},\omega)$ and $\tilde{\mathbf{J}}(\mathbf{r},\omega)$ are contained in a small region of space and $S \to \infty$,

$$\tilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

Electromagnetic waves from time harmonic sources – continued:

For scalar potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorentz gauge, $k \equiv \frac{\omega}{c}$)

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

04/24/2015 PHY 712 Spring 2015 -- Lecture 36 22

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

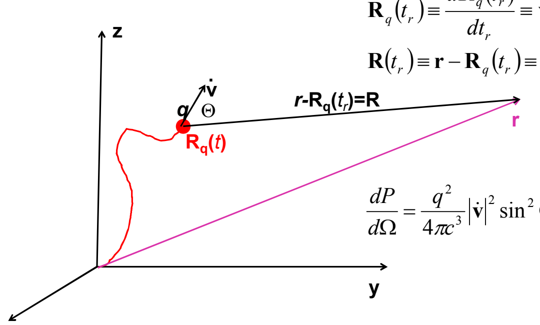
Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm}\tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r' \tilde{\rho}(\mathbf{r}',\omega) j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

04/24/2015 PHY 712 Spring 2015 -- Lecture 36 23

Radiation from a moving charged particle



Variables (notation) :

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3}|\dot{\mathbf{v}}|^2 \sin^2\Theta$$

04/24/2015 PHY 712 Spring 2015 -- Lecture 36 24

Liènard-Wiechert potentials –(Gaussian units)

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\left(1 - \frac{v^2}{c^2}\right) + \left(\mathbf{R}\times\left\{\left(\mathbf{R} - \frac{\mathbf{v}R}{c}\right)\times\frac{\dot{\mathbf{v}}}{c^2}\right\}\right)\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{q}{c}\left[\frac{-\mathbf{R}\times\mathbf{v}}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^3}\left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}}\cdot\mathbf{R}}{c^2}\right) - \frac{\mathbf{R}\times\dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v}\cdot\mathbf{R}}{c}\right)^2}\right]$$

$$\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{R}\times\mathbf{E}(\mathbf{r},t)}{R}.$$

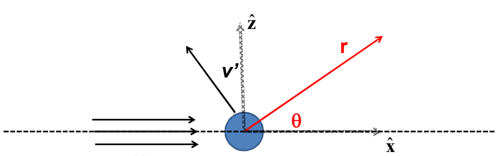
Example:

Problem 15.2 in Jackson:

A nonrelativistic particle of charge *e* and mass *m* collides with a fixed, smooth, hard sphere of radius *R*. Assuming that the collision is elastic, show that in the dipole approximation (neglecting retardation effects) the classical differential cross section for the emission of photons per unit solid angle per unit energy interval is:

$$\frac{d^2\sigma}{d\Omega d(\hbar\omega)} = \frac{R^2}{12\pi}\frac{e^2}{\hbar c}\left(\frac{v}{c}\right)^2\frac{1}{\hbar\omega}\left(2 + 3\sin^2\theta\right)$$

where θ is measured relative to the incident direction.



Suppose that

$\mathbf{v} = v\hat{\mathbf{z}}$

$\mathbf{v}' = v(\sin a \cos b\,\hat{\mathbf{x}} + \sin a \sin b\,\hat{\mathbf{y}} + \cos a\,\hat{\mathbf{z}})$

$\mathbf{r} = \sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}$

$\boldsymbol{\varepsilon}_1 = \hat{\mathbf{y}}$

$\boldsymbol{\varepsilon}_2 = -\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{z}}$

Cross section depends on $\left\langle \left|\boldsymbol{\varepsilon}_i\cdot(\mathbf{v}' - \mathbf{v})\right|^2\right\rangle$

Low frequency radiation from charged particle during a collision as analyzed by Eq. 15.2 :

$$\frac{d^2I}{d\omega d\Omega} = \frac{e^2}{4\pi^2 c}\left|\boldsymbol{\varepsilon}\cdot\left(\frac{\boldsymbol{\beta}'}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}'} - \frac{\boldsymbol{\beta}}{1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta}}\right)\right|^2 \approx \frac{e^2}{4\pi^2 c^3}\left|\boldsymbol{\varepsilon}\cdot(\mathbf{v}'-\mathbf{v})\right|^2$$

$$\frac{d^2\sigma}{d\Omega d(\hbar\omega)} = \left\langle \frac{d^2 I}{d\omega d\Omega}\frac{d\sigma}{d\Omega}\right\rangle = \frac{e^2}{4\pi^2 c^3}\frac{R^2}{4}\left\langle \left|\boldsymbol{\varepsilon}\cdot(\mathbf{v}'-\mathbf{v})\right|^2\right\rangle$$

For: $\mathbf{v} = v\hat{\mathbf{z}} \qquad \mathbf{v}' = v(\sin a\cos b\hat{\mathbf{x}} + \sin a \sin b\hat{\mathbf{y}} + \cos a\hat{\mathbf{z}})$

$$\mathbf{v}'-\mathbf{v} = v(\sin a\cos b\hat{\mathbf{x}} + \sin a\sin b\hat{\mathbf{y}} + (\cos a - 1)\hat{\mathbf{z}})$$

$$\boldsymbol{\varepsilon}_1 = \hat{\mathbf{y}} \qquad \Rightarrow \left\langle \left|\boldsymbol{\varepsilon}_1\cdot(\mathbf{v}'-\mathbf{v})\right|^2\right\rangle = v^2\int d\cos b\, da(\sin a\sin b)^2 = \frac{4\pi}{3}v^2$$

$$\boldsymbol{\varepsilon}_2 = -\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}$$

$$\left\langle \left|\boldsymbol{\varepsilon}_2\cdot(\mathbf{v}'-\mathbf{v})\right|^2\right\rangle = v^2\int d\cos b\, da(-\cos\theta\sin a\cos b + \sin\theta(\cos a - 1))^2 = \frac{4\pi}{3}v^2(\cos^2\theta + 4\sin^2\theta)$$

$$\frac{d^2\sigma}{d\Omega d(\hbar\omega)} = \frac{e^2 v^2 R^2}{12\pi c^3}(1+\cos^2\theta + 4\sin^2\theta)$$

04/24/2015 PHY 712 Spring 2015 -- Lecture 36 28