PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

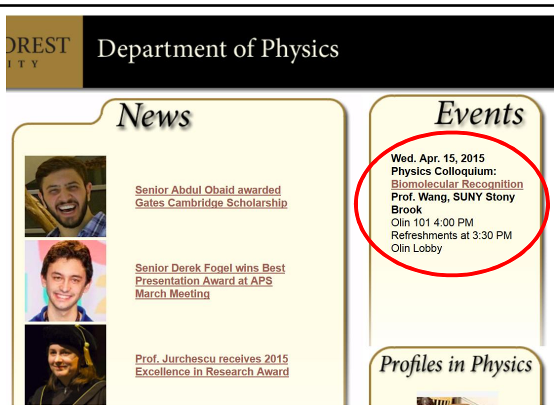
Plan for Lecture 32:

Special Topics in Electrodynamics:

Cherenkov radiation

References: Jackson Chapter 13.4
Zangwill Chapter 23.7
Smith Chapter 6.4

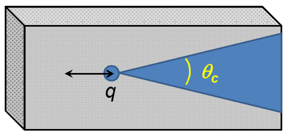
20	Mon: 03/16/2015	Chap. 8	Review Exam; Wave guides	#19	03/18/2015
21	Wed: 03/18/2015	Chap. 8	Wave guides	#20	03/20/2015
22	Fri: 03/20/2015	Chap. 9	Radiation sources	#21	03/23/2015
23	Mon: 03/23/2015	Chap. 9 & 10	Radiation and scattering	#22	03/25/2015
24	Wed: 03/25/2015	Chap. 9 & 10	Radiation and scattering		
25	Fri: 03/27/2015	Chap. 11	Special relativity	#23	03/30/2015
26	Mon: 03/30/2015	Chap. 11	Special relativity	#24	04/01/2015
27	Wed: 04/01/2015	Chap. 11	Special relativity	#25	04/06/2015
	Fri: 04/03/2015	Good Friday	No class		
28	Mon: 04/06/2015	Chap. 14	Radiation from moving charges	#26	04/08/2015
29	Wed: 04/08/2015	Chap. 14	Radiation from moving charges	#27	04/10/2015
30	Fri: 04/10/2015	Chap. 14	Radiation from moving charges	#28	04/13/2015
31	Mon: 04/13/2015	Chap. 15	Radiation due to scattering	#29	04/15/2015
32	Wed: 04/16/2015	Chap. 13	Cherenkov radiation	#30	04/17/2015
33	Fri: 04/17/2015		Special topics -- superconductivity		04/20/2015
34	Mon: 04/20/2015		Special topics -- superconductivity		
35	Wed: 04/22/2015		Review		
36	Fri: 04/24/2015		Review		
	Mon: 04/27/2015		Presentations I		
	Wed: 04/29/2015		Presentations II		
	Fri: 05/01/2015		Presentations III & Take home exam		



References for notes: Glenn S. Smith, *An Introduction to Electromagnetic Radiation* (Cambridge UP, 1997), Andrew Zangwill, Modern Electrodynamics (Cambridge UP, 2013)

Cherenkov radiation

Discovered ~1930; bluish light emitted by energetic charged particles traveling within dielectric materials



Maxwell's potential equations within a material having permittivity and permeability (Lorentz gauge; cgs Gaussian units)

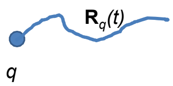
$$\nabla^2\Phi - \mu\varepsilon\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{4\pi}{\varepsilon}\rho$$

$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi\mu}{c}\mathbf{J}$$

Source: charged particle moving on trajectory $\mathbf{R}_q(t)$:

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{R}_q(t))$$

$$\mathbf{J}(\mathbf{r},t) = q\dot{\mathbf{R}}_q(t)\delta(\mathbf{r}-\mathbf{R}_q(t))$$



Liénard-Wiechert potential solutions:

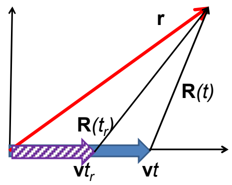
$$\Phi(\mathbf{r},t) = \frac{q}{\varepsilon}\frac{1}{R(t_r) - \boldsymbol{\beta}_n\cdot\mathbf{R}(t_r)}$$

$$\mathbf{A}(\mathbf{r},t) = q\mu\frac{\boldsymbol{\beta}_n}{R(t_r) - \boldsymbol{\beta}_n\cdot\mathbf{R}(t_r)}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$\boldsymbol{\beta}_n(t_r) \equiv \frac{\dot{\mathbf{R}}_q(t_r)}{c_n} \qquad c_n \equiv \sqrt{\mu\varepsilon}\, c \equiv \frac{c}{n}$$

$$t_r = t - \frac{R(t_r)}{c_n}$$



Some algebra

$$\mathbf{R}(t)=\mathbf{r}-\mathbf{v}t$$

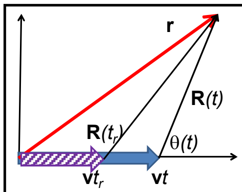
$$\mathbf{R}(t_r)=\mathbf{r}-\mathbf{v}t_r=\mathbf{R}(t)+\mathbf{v}(t-t_r)$$

$$(t-t_r)c_n=R(t_r)=\left|\mathbf{R}(t)+\mathbf{v}(t-t_r)\right|$$

Quadratic equation for $(t-t_r)c_n$:

$$\left((t-t_r)c_n\right)^2=R^2(t)+2\mathbf{R}(t)\cdot\boldsymbol{\beta}_n(t-t_r)c_n+\beta_n^{\,2}\left((t-t_r)c_n\right)^2$$

$$(t-t_r)c_n=\frac{-\mathbf{R}(t)\cdot\boldsymbol{\beta}_n\pm\sqrt{\left(\mathbf{R}(t)\cdot\boldsymbol{\beta}_n\right)^2-\left(\beta_n^{\,2}-1\right)R^2(t)}}{\beta_n^{\,2}-1}$$



$$\mathbf{R}(t_r)=\mathbf{r}-\mathbf{v}t_r=\mathbf{R}(t)+\mathbf{v}(t-t_r)$$

$$(t-t_r)c_n=R(t_r)$$

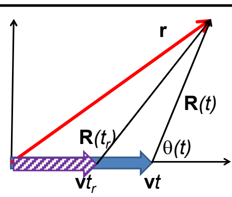
$$R(t_r)-\mathbf{R}(t_r)\cdot\boldsymbol{\beta}_n=(t-t_r)c_n\left(1-\beta_n^2\right)-\mathbf{R}(t)\cdot\boldsymbol{\beta}_n$$

$$R(t_r)=\frac{-\mathbf{R}(t)\cdot\boldsymbol{\beta}_n\pm\sqrt{\left(\mathbf{R}(t)\cdot\boldsymbol{\beta}_n\right)^2-\left(\beta_n^{\,2}-1\right)R^2(t)}}{\beta_n^{\,2}-1}$$

$$R(t_r)=\frac{R(t)}{\beta_n^{\,2}-1}\left(-\beta_n\cos\theta\pm\sqrt{1-\beta_n^{\,2}\sin^2\theta}\right)$$

$$R(t_r)-\mathbf{R}(t_r)\cdot\boldsymbol{\beta}_n=\mp R(t)\sqrt{1-\beta_n^{\,2}\sin^2\theta}$$

Liénard-Wiechert potentials:



$$\Phi(\mathbf{r},t)=\pm\frac{q}{\varepsilon}\frac{1}{R(t)\sqrt{1-\beta_n^{\,2}\sin^2\theta}}$$

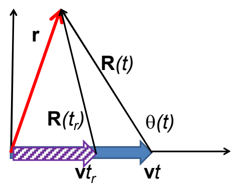
$$\mathbf{A}(\mathbf{r},t)=\pm q\mu\frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n^{\,2}\sin^2\theta}}$$

For $\beta_n>1$, range of θ is limited:

$$R(t_r)=\frac{R(t)}{\beta_n^{\,2}-1}\left(-\beta_n\cos\theta\pm\sqrt{1-\beta_n^{\,2}\sin^2\theta}\right)\geq 0$$

$$\Rightarrow\frac{\pi}{2}\leq\theta\leq\sin^{-1}\left(\frac{1}{\beta_n}\right)$$

Physical fields for $\beta_n > 1$



$$\frac{\pi}{2} \le \theta \le \sin^{-1}\left(\frac{1}{\beta_n}\right)$$

$$\text{Define } \cos\theta_C \equiv \sqrt{1-\frac{1}{\beta_n^2}}$$

$$\Rightarrow \cos\theta \le \cos\theta_C$$

$$\Phi(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{1}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r},t) = 2q\mu\frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

Physical fields for $\beta > 1$

$$\Phi(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{1}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{A}(\mathbf{r},t) = 2q\mu\frac{\boldsymbol{\beta}_n}{R(t)\sqrt{1-\beta_n^2\sin^2\theta}}\Theta(\cos\theta_C - \cos\theta(t))$$

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi - \frac{1}{c_n}\frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{\varepsilon}\frac{\hat{\mathbf{R}}}{(R(t))^2\sqrt{1-\beta_n^2\sin^2\theta}} \times$$

$$\left(\frac{\beta_n^2-1}{1-\beta_n^2\sin^2\theta}\Theta(\cos\theta_C - \cos\theta(t)) + \sqrt{\beta_n^2-1}\;\delta(\cos\theta_C - \cos\theta(t))\right)$$

$$\mathbf{B}(\mathbf{r},t) = -\beta_n\sin\theta\left(\hat{\theta}\times\mathbf{E}(\mathbf{r},t)\right)$$

Intermediate steps:

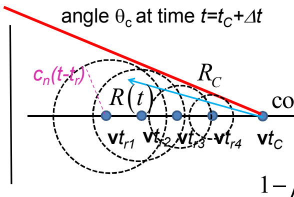
$$\frac{d\theta}{dt} = \frac{v\sin\theta}{R} \qquad \frac{dR}{dt} = -v\cos\theta$$

Using instantaneous polar coordinates: $\nabla \equiv \hat{\mathbf{R}}\frac{\partial}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial}{\partial\theta}$

$$\nabla\Theta(\cos\theta_C - \cos\theta(t)) = \delta(\cos\theta_C - \cos\theta(t))\frac{\sin\theta(t)}{R(t)}\hat{\boldsymbol{\theta}}$$

$$\frac{\partial\Theta(\cos\theta_C - \cos\theta(t))}{\partial t} = \delta(\cos\theta_C - \cos\theta(t))\frac{v\sin^2\theta(t)}{R(t)}$$

Some details: Cherenkov radiation observed near the angle θ_c at time $t=t_C+\Delta t$



$$\cos\theta_C - \cos\theta(t) \approx \frac{c_n \Delta t}{\beta_n R_C}$$

$$1-\beta_n \sin^2\theta(t) \approx \frac{2c_n \Delta t\sqrt{\beta_n^2-1}}{R_C}$$

When the dust clears ....

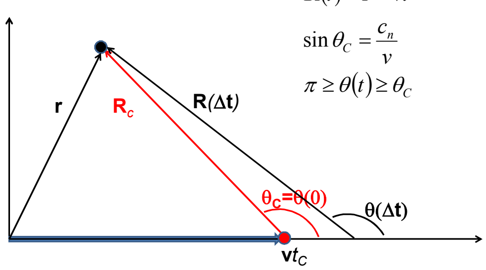
$$\frac{d^2 I}{d\omega d\ell} \propto \left(1-\frac{c_n^2}{v^2}\right)\omega$$

Some details: Cherenkov radiation observed near the angle θ_c

$$\mathbf{R}(t) = \mathbf{r} - \mathbf{v}t$$

$$\sin\theta_C = \frac{c_n}{v}$$

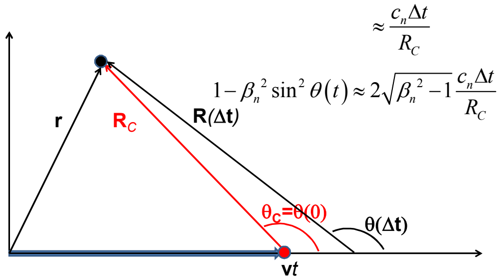
$$\pi \geq \theta(t) \geq \theta_C$$



Cherenkov radiation observed near the angle θ_c -- continued

$$\cos\theta_C - \cos\theta(\Delta t) \approx \sin\theta_C \; \theta(\Delta t)$$
$$\approx \frac{c_n \Delta t}{R_C}$$

$$1-\beta_n^2 \sin^2\theta(t) \approx 2\sqrt{\beta_n^2-1}\,\frac{c_n\Delta t}{R_C}$$



Cherenkov radiation observed near the angle θ_c -- continued

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{\varepsilon} \frac{\hat{\mathbf{R}}}{(R(t))^2 \sqrt{1-\beta_n^2 \sin^2\theta}} \times$$
$$\left(\frac{\beta_n^2 - 1}{1-\beta_n^2 \sin^2\theta} \Theta(\cos\theta_C - \cos\theta(t)) + \sqrt{\beta_n^2 - 1}\, \delta(\cos\theta_C - \cos\theta(t)) \right)$$

$$\mathbf{B}(\mathbf{r},t) = -\beta_n \sin\theta \left(\hat{\theta} \times \mathbf{E}(\mathbf{r},t)\right)$$

Estimates at $t = t_C + \Delta t$

$$\mathbf{E}(\mathbf{r},t) \approx -\frac{2q}{\varepsilon} \hat{\mathbf{R}}_C \frac{\left(\beta_n^2 - 1\right)^{1/4}}{\left(2c_n^3 R_C\right)^{1/2}} \left[(\Delta t)^{-1/2} \delta(\Delta t) - \frac{1}{2} (\Delta t)^{-3/2} \Theta(\Delta t) \right]$$

$$\mathbf{B}(\mathbf{r},t) = -\left(\hat{\boldsymbol{\theta}}(0) \times \mathbf{E}(\mathbf{r},t)\right)$$

Cherenkov radiation observed near the angle θ_c -- continued

Spectral analysis:

$$\tilde{\mathbf{E}}(\omega) = -\frac{2q}{\varepsilon} \hat{\mathbf{R}}_C \frac{\left(\beta_n^2 - 1\right)^{1/4}}{\left(2c_n^3 R_C\right)^{1/2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \left[t^{-1/2} \delta(t) - \frac{1}{2} t^{-3/2} \Theta(t) \right] e^{i\omega t}$$

$$= -i\omega \frac{2q}{\varepsilon} \hat{\mathbf{R}}_C \frac{\left(\beta_n^2 - 1\right)^{1/4}}{\left(2c_n^3 R_C\right)^{1/2}} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dt \; t^{-1/2} e^{i\omega t}$$

$$= \frac{q}{\varepsilon} \hat{\mathbf{R}}_C \frac{\left(\beta_n^2 - 1\right)^{1/4}}{\left(2c_n^3 R_C\right)^{1/2}} (1-i)\sqrt{\omega}$$

Spectral intensity: $$\frac{d^2 I}{d\Omega d\omega} \propto \left|\tilde{\mathbf{E}}(\omega)\right|^2 = \frac{q^2}{\varepsilon^2} \frac{\left(\beta_n^2 - 1\right)^{1/2}}{c_n^3 R_C} \omega$$

Cherenkov radiation emitted by the core of the Reed Research Reactor located at Reed College in Portland, Oregon, U.S. *Cherenkov radiation*. Photograph. *Encyclopædia Britannica Online*. Web. 12 Apr. 2013. http://www.britannica.com/EBchecked/media/174732