

**PHY 712 Electrodynamics  
9-9:50 AM MWF Olin 103**

**Plan for Lecture 28:**

**Continue reading Chap. 14 –  
Radiation by moving charges**

- 1. Motion in a line**
- 2. Motion in a circle**
- 3. Spectral analysis of radiation**

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Week	Date	Topic	Notes	Exam
20	Mon: 03/16/2015	Chap. 8	Review Exam: Wave guides	#19 03/18/2015
21	Wed: 03/18/2015	Chap. 8	Wave guides	#20 03/20/2015
22	Fri: 03/20/2015	Chap. 9	Radiation sources	#21 03/23/2015
23	Mon: 03/23/2015	Chap. 9 & 10	Radiation and scattering	#22 03/25/2015
24	Wed: 03/25/2015	Chap. 9 & 10	Radiation and scattering	
25	Fri: 03/27/2015	Chap. 11	Special relativity	#23 03/30/2015
26	Mon: 03/30/2015	Chap. 11	Special relativity	#24 04/01/2015
27	Wed: 04/01/2015	Chap. 11	Special relativity	#25 04/06/2015
	Fri: 04/03/2015	Good Friday	No class	
29	Mon: 04/06/2015	Chap. 14	Radiation from moving charges	#26 04/08/2015
29	Wed: 04/08/2015	Chap. 14	Radiation from moving charges	#27 04/10/2015
30	Fri: 04/10/2015			04/13/2015
31	Mon: 04/13/2015			04/15/2015
32	Wed: 04/15/2015			04/17/2015
33	Fri: 04/17/2015			04/20/2015
34	Mon: 04/20/2015			
35	Wed: 04/22/2015			
36	Fri: 04/24/2015			
	Mon: 04/27/2015		Presentations I	
	Wed: 04/29/2015		Presentations II	
	Fri: 05/01/2015		Presentations III & Take home exam	

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**Radiation from a moving charged particle**

Variables (notation) :

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

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Liénard-Wiechert fields (cgs Gaussian units):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})^3} \left[ \left( \mathbf{R} - \frac{\mathbf{v}R}{c} \right) \left( 1 - \frac{v^2}{c^2} \right) + \left( \mathbf{R} \times \left\{ \left( \mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right]. \quad (19)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{c} \left[ \frac{-\mathbf{R} \times \mathbf{v}}{(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})^3} \left( 1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c})^2} \right]. \quad (20)$$

In this case, the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}. \quad (21)$$

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v} \quad \mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R} \quad \dot{\mathbf{v}} \equiv \frac{d^2\mathbf{R}_q(t_r)}{dt_r^2}$$

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Electric field far from source:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{\left( R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^3} \left\{ \mathbf{R} \times \left[ \left( \mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{R}$$

$$\text{Let } \hat{\mathbf{R}} \equiv \frac{\mathbf{R}}{R} \quad \boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} \quad \dot{\boldsymbol{\beta}} \equiv \frac{\dot{\mathbf{v}}}{c}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \left\{ \hat{\mathbf{R}} \times \left[ (\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)$$

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Poynting vector:

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B})$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{cR(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \left\{ \hat{\mathbf{R}} \times \left[ (\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right] \right\}$$

$$\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{R}} \times \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \hat{\mathbf{R}} |\mathbf{E}(\mathbf{r}, t)|^2 = \frac{q^2}{4\pi c R^2} \hat{\mathbf{R}} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

Note: We have assumed that

$$\hat{\mathbf{R}} \cdot \mathbf{E}(\mathbf{r}, t) = 0$$

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Power radiated

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \hat{\mathbf{R}} |\mathbf{E}(\mathbf{r}, t)|^2 = \frac{q^2}{4\pi c R^2} \hat{\mathbf{R}} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6}$$

In the non-relativistic limit:  $\beta \ll 1$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c} |\hat{\mathbf{R}} \times [\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}}]|^2 = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

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Radiation from a moving charged particle

Variables (notation):

$$\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

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Radiation power in non-relativistic case -- continued

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \sin^2 \Theta$$

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2$$

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Radiation distribution in the relativistic case

$$\frac{dP}{d\Omega} = \mathbf{S} \cdot \hat{\mathbf{R}} R^2 = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^6} \right|_{t_r = t - R/c}$$

This expression gives us the energy per unit field time  $t$ . We are often interested in the power per unit retarded time  $t_r = t - R/c$ :

$$\frac{dP(t)}{d\Omega} = \frac{dP_r(t_r)}{d\Omega} \frac{dt_r}{dt} \quad \frac{dt}{dt_r} = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}}$$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|_{t_r = t - R/c}$$

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Radiation distribution in the relativistic case -- continued

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|_{t_r = t - R/c}$$

For linear acceleration:  $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times (\dot{\boldsymbol{\beta}} \times \hat{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

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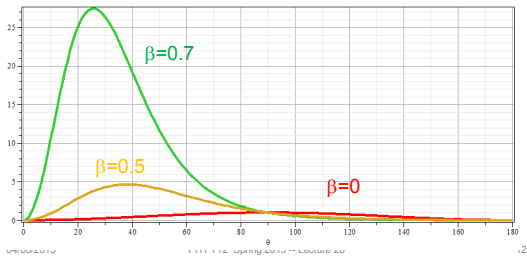
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Power from linearly accelerating particle

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \left. \frac{|\hat{\mathbf{R}} \times (\dot{\boldsymbol{\beta}} \times \hat{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \right|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$




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Power from linearly accelerating particle

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6 \quad \text{where } \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

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Power distribution for linear acceleration -- continued

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \dot{\boldsymbol{\beta}})|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c} = \frac{q^2}{4\pi c^3} |\dot{\mathbf{v}}|^2 \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$

$$P_r(t_r) = \int \frac{dP_r(t_r)}{d\Omega} d\Omega = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^6 \quad \text{where } \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}$$

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Power distribution for circular acceleration

$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c} \frac{|\dot{\boldsymbol{\beta}}|^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2 - (\hat{\mathbf{R}} \cdot \dot{\boldsymbol{\beta}})^2 (1 - \beta^2)}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^5} \bigg|_{t_r = t - R/c}$$

$$P_r(t_r) = \int d\Omega \frac{dP_r(t_r)}{d\Omega} = \frac{2}{3} \frac{q^2}{c^3} |\dot{\mathbf{v}}|^2 \gamma^4$$

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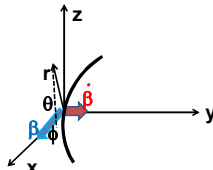
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Power distribution for circular acceleration



$$\frac{dP_r(t_r)}{d\Omega} = \frac{q^2}{4\pi c} \frac{|\dot{\beta}|^2 (1 - \beta \cdot \hat{R})^2 - (\hat{R} \cdot \dot{\beta})^2 (1 - \beta^2)}{(1 - \beta \cdot \hat{R})^5} \Bigg|_{t_r = t - R/c}$$

$$= \frac{q^2}{4\pi c^3} \frac{|\dot{v}|^2}{(1 - \beta \cos(\theta))^3} \left( 1 - \frac{\cos^2 \theta \sin^2 \phi}{\gamma^2 (1 - \beta \cos(\theta))^2} \right)$$

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Spectral composition of electromagnetic radiation  
Previously we determined the power distribution from a charged particle:

$$\frac{dP(t)}{d\Omega} = \mathbf{S} \cdot \hat{R} R^2 = \frac{q^2}{4\pi} \frac{|\hat{R} \times [(\hat{R} - \beta) \times \dot{\beta}]|^2}{(1 - \beta \cdot \hat{R})^6} \Bigg|_{t_r = t - R/c}$$

$$\equiv |\mathbf{a}(t)|^2$$

where  $\mathbf{a}(t) \equiv \sqrt{\frac{q^2}{4\pi}} \frac{|\hat{R} \times [(\hat{R} - \beta) \times \dot{\beta}]|}{(1 - \beta \cdot \hat{R})^3} \Bigg|_{t_r = t - R/c}$

Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

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Spectral composition of electromagnetic radiation -- continued

Time integrated power per solid angle :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Fourier amplitude :

$$\tilde{\mathbf{a}}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t} \quad \mathbf{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{\mathbf{a}}(\omega) e^{-i\omega t}$$

Parseval's theorem  
**Marc-Antoine Parseval des Chênes 1755-1836**  
<http://www-history.mcs.st-andrews.ac.uk/Biographies/Parseval.html>

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Spectral composition of electromagnetic radiation -- continued

Consequences of Parseval's analysis :

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP(t)}{d\Omega} = \int_{-\infty}^{\infty} dt |\mathbf{a}(t)|^2 = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2$$

Note that:  $\tilde{\mathbf{a}}(\omega) = \tilde{\mathbf{a}}^*(-\omega)$

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} d\omega |\tilde{\mathbf{a}}(\omega)|^2 = \int_0^{\infty} d\omega \left( |\tilde{\mathbf{a}}(\omega)|^2 + |\tilde{\mathbf{a}}(-\omega)|^2 \right) \equiv \int_0^{\infty} d\omega \frac{\partial^2 I}{\partial \Omega \partial \omega}$$

$$\frac{\partial^2 I}{\partial \Omega \partial \omega} \equiv 2 |\tilde{\mathbf{a}}(\omega)|^2$$

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Spectral composition of electromagnetic radiation -- continued

For our case:  $\mathbf{a}(t) \equiv \sqrt{\frac{q^2}{4\pi c}} \left. \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c}$

Fourier amplitude :

$$\begin{aligned} \tilde{\mathbf{a}}(\omega) &\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \left. \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c} e^{i\omega t} \end{aligned}$$

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Spectral composition of electromagnetic radiation -- continued

Fourier amplitude :

$$\begin{aligned} \tilde{\mathbf{a}}(\omega) &\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \mathbf{a}(t) e^{i\omega t} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt \left. \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c} e^{i\omega t} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{dt}{dt_r} \left. \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)} \\ &= \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \left. \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^3} \right|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)} \end{aligned}$$

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Spectral composition of electromagnetic radiation -- continued

Exact expression :

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Recall:  $\dot{\mathbf{R}}_q(t_r) \equiv \frac{d\mathbf{R}_q(t_r)}{dt_r} \equiv \mathbf{v}$     $\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r) \equiv \mathbf{R}$

For  $r \gg R_q(t_r)$     $R(t_r) \approx r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)$    where    $\hat{\mathbf{r}} \equiv \frac{\mathbf{r}}{r}$

At the same level of approximation :  $\hat{\mathbf{R}} \approx \hat{\mathbf{r}}$

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Spectral composition of electromagnetic radiation -- continued

Exact expression:

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{R}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r + R(t_r)/c)}$$

Approximate expression:

$$\tilde{\mathbf{a}}(\omega) = \sqrt{\frac{q^2}{8\pi^2 c}} e^{i\omega(r/c)} \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)}$$

Resulting spectral intensity expression:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

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Example – radiation from a collinear acceleration burst

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \frac{\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} \Bigg|_{t_r = t - R/c} e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

Suppose that  $\boldsymbol{\beta} = \begin{cases} \frac{\hat{\boldsymbol{\beta}} \Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left| \frac{\hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \hat{\boldsymbol{\beta}}] \Delta v}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2 \tau} \int_0^\tau dt_r e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \boldsymbol{\beta} \tau)} \right|^2 \quad \text{Let } \boldsymbol{\beta} \cdot \hat{\mathbf{r}} = \beta \cos \theta$$

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left( \frac{\Delta v \sin \theta}{(1 - \beta \cos \theta)^2} \frac{\sin(\omega\tau(1 - \beta \cos \theta)/2)}{(\omega\tau(1 - \beta \cos \theta)/2)} \right)^2$$

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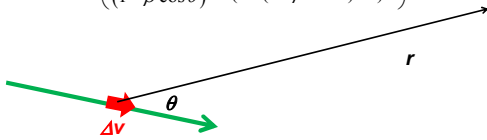
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Example:

$$\text{Suppose that } \dot{\boldsymbol{\beta}} = \begin{cases} \frac{\hat{\boldsymbol{\beta}} \Delta v}{c\tau} & 0 < t_r < \tau \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2}{4\pi^2 c^3} \left( \frac{\Delta v \sin \theta}{(1 - \beta \cos \theta)^2}, \frac{\sin(\omega\tau(1 - \beta \cos \theta)/2)}{(\omega\tau(1 - \beta \cos \theta)/2)} \right)^2$$



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Spectral composition of electromagnetic radiation -- continued

Alternative expression --

It can be shown that:

$$\frac{\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2} = \frac{d}{dt_r} \left( \frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta})}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})} \right)$$

Integration by parts and assumptions about the integration limit behavior shows that the spectral intensity depends on the following integral:

$$\frac{\partial^2 I}{\partial \omega \partial \Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int_{-\infty}^{\infty} dt_r \left[ \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \boldsymbol{\beta}(t_r)) \right] e^{i\omega(t_r - \hat{\mathbf{r}} \cdot \mathbf{R}_q(t_r)/c)} \right|^2$$

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