PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 23:

Continue reading Chap. 9 & 10

A. **Electromagnetic waves due to specific sources**

B. **Dipole radiation examples**

C. **Scattered radiation**

12	Wed: 02/11/2015	Chap. 5	Magnetostatics	#12	02/13/2015
13	Fri: 02/13/2015	Chap. 5	Magnetostatics	#13	02/16/2015
14	Mon: 02/16/2015	Chap. 6	Maxwell's equations	#14	02/18/2015
15	Wed: 02/18/2015	Chap. 6	Electromagnetic energy and force	#15	02/20/2015
16	Fri: 02/20/2015	Chap. 7	Electromagnetic plane waves	#16	02/23/2015
17	Mon: 02/23/2015	Chap. 7	Dielectric media	#17	02/25/2015
18	Wed: 02/25/2015	Chap. 7	Complex dielectrics	#18	02/27/2015
19	Fri: 02/27/2015	Chap. 1-7	Review -- Take home exam distributed		
	Mon. 03/02/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Wed. 03/04/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Fri. 03/06/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Mon. 03/09/2015	*Spring Break*			
	Wed. 03/11/2015	*Spring Break*			
	Fri. 03/13/2015	*Spring Break*			
20	Mon: 03/16/2015	Chap. 8	Review Exam; Wave guides	#19	03/18/2015
21	Wed: 03/18/2015	Chap. 8	Wave guides	#20	03/20/2015
22	Fri: 03/20/2015	Chap. 9	Radiation sources	#21	03/23/2015
23	Mon: 03/23/2015	Chap. 9 & 10	Radiation and scattering	#22	03/25/2015

Review:

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$

Ampere-Maxwell's law: $\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \frac{1}{\varepsilon_0 \mu_0}$$

Review:

Formulation of Maxwell's equations in terms of vector and scalar potentials:

Lorentz gauge form -- require : $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Note that the Lorentz gauge is consistent with the source continuity condition : $\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r},t) = 0$

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Review:

Electromagnetic waves from time harmonic sources

Charge density : $\rho(\mathbf{r},t) = \Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Current density : $\mathbf{J}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

$\Rightarrow$ Scalar potential : $\Phi(\mathbf{r},t) = \Re\left(\tilde{\Phi}(\mathbf{r},\omega)e^{-i\omega t}\right)$

$\Rightarrow$ Vector potential : $\mathbf{A}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{A}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

For $k \equiv \frac{\omega}{c}$:

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi}\int d^3r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

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Review:

Electromagnetic waves from time harmonic sources – continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm}\tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r' \tilde{\rho}(\mathbf{r}',\omega) j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm}\tilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r' \tilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

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Review:

Forms of spherical Bessel and Hankel functions:

$$j_0(x)=\frac{\sin(x)}{x} \qquad h_0(x)=\frac{e^{ix}}{ix}$$

$$j_1(x)=\frac{\sin(x)}{x^2}-\frac{\cos(x)}{x} \qquad h_1(x)=-\left(1+\frac{i}{x}\right)\frac{e^{ix}}{x}$$

$$j_2(x)=\left(\frac{3}{x^3}-\frac{1}{x}\right)\sin(x)-\frac{3\cos(x)}{x^2} \qquad h_2(x)=i\left(1+\frac{3i}{x}-\frac{3}{x^2}\right)\frac{e^{ix}}{x}$$

Assymptotic behavior :

$$x<<1 \quad \Rightarrow j_l(x)\approx\frac{(x)^l}{(2l+1)!!}$$

$$x>>1 \quad \Rightarrow h_l(x)\approx(-i)^{l+1}\frac{e^{ix}}{x}$$

Review:

Electromagnetic waves from time harmonic sources – continued:

Dipole radiation case :

Define dipole moment at frequency ω :

$$\mathbf{p}(\omega)\equiv\int d^3r\,\mathbf{r}\tilde{\rho}(\mathbf{r},\omega)=-\frac{1}{i\omega}\int d^3r\,\tilde{\mathbf{J}}(\mathbf{r},\omega)$$

For fields outside extent of source and $kr'<<1$ within the source

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=-\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{i}{4\pi\omega\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Review:

Electromagnetic waves from time harmonic sources – continued:

$$\tilde{\mathbf{E}}(\mathbf{r},\omega)=-\nabla\tilde{\Phi}(\mathbf{r},\omega)+i\omega\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2((\hat{\mathbf{r}}\times\mathbf{p}(\omega))\times\hat{\mathbf{r}})+\left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega))-\mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega)=\nabla\times\tilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$=\frac{1}{4\pi\varepsilon_0c^2}\frac{e^{ikr}}{r}k^2(\hat{\mathbf{r}}\times\mathbf{p}(\omega))\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr>>1$:

$$\frac{dP}{d\Omega}=r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg}=\frac{r^2}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\tilde{\mathbf{E}}(\mathbf{r},\omega)\times\tilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$=\frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|(\hat{\mathbf{r}}\times\mathbf{p}(\omega))\times\hat{\mathbf{r}}\right|^2$$

Properties of dipole radiation field for *kr* >>1:

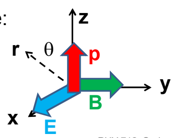
$$\widetilde{\mathbf{E}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right)\right)$$

$$\widetilde{\mathbf{B}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega}=r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg}=\frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

Example:



Alternative approach

Fields from time harmonic source:

$$\tilde{\Phi}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

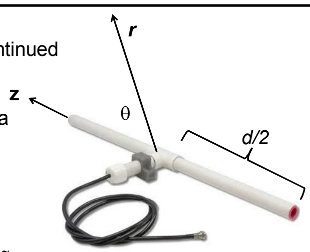
For $r >> r'$: $\quad |\mathbf{r}-\mathbf{r}'|\approx r\left(1-\hat{\mathbf{r}}\cdot\mathbf{r}'+...\right)$

$$\tilde{\Phi}(\mathbf{r},\omega)\approx\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Alternative approach -- continued

Linear center-fed antenna



$$\tilde{\mathbf{A}}(\mathbf{r},\omega)\approx$$

$$\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3r'\, e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{J}}(\mathbf{r}',\omega)=I_0\sin\left(\frac{kd}{2}-k|z|\right)\delta(x)\delta(y)\hat{\mathbf{z}}$$

Alternative approach – linear center-fed antenna continued

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) \approx \hat{\mathbf{z}} \frac{\mu_0 I_0}{4\pi} \frac{e^{ikr}}{r} \int_{-d/2}^{d/2} dz'\, e^{-ik\cos(\theta)z'} \sin\left(\frac{kd}{2} - k|z'|\right)$$

$$= \hat{\mathbf{z}} \frac{\mu_0 I_0}{2\pi} \frac{e^{ikr}}{r} \left(\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta} \right)$$

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

Alternative approach – linear center-fed antenna continued

Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

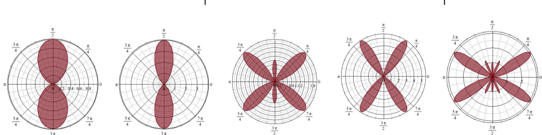
for $kd = \pi$: $$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{8\pi^2} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

for $kd = 2\pi$: $$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{4}{8\pi^2} \frac{\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}$$

Alternative approach – linear center-fed antenna continued

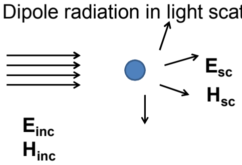
Time averaged power:

$$\frac{dP}{d\Omega} = I_0^2 \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$



kd=π 2π 3π 4π 5π

Dipole radiation in light scattering by small (dielectric) particles

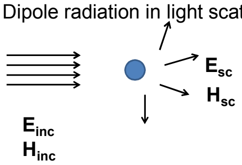


$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0\cdot\mathbf{r}} \qquad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left((\hat{\mathbf{r}}\times\mathbf{p})\times\hat{\mathbf{r}}\right) \qquad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

Dipole radiation in light scattering by small (dielectric) particles



Scattering cross section :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}_{sc}\rangle_{avg}}{\hat{\mathbf{k}}_0\cdot\langle\mathbf{S}_{inc}\rangle_{avg}}$$

$$= \frac{r^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0\cdot\mathbf{E}_{inc}\right|^2} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\right|^2$$

Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity ε and radius *a*:

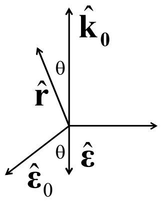
$$\mathbf{p} = 4\pi a^3\varepsilon_0\left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)\mathbf{E}_{inc}$$

Scattering cross section :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{r^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0\cdot\mathbf{E}_{inc}\right|^2} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\right|^2$$

$$= k^4 a^6\left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

Scattering by dielectric sphere with permittivity ε and radius *a*:

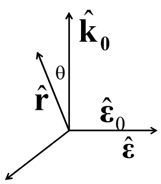


For $\mathbf{E}_{inc}$ polarized in scattering plane:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) == k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left|\hat{\boldsymbol{\varepsilon}}\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$= k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \cos^2\theta$$

Scattering by dielectric sphere with permittivity ε and radius *a*:



For $\mathbf{E}_{inc}$ polarized perpendicular to scattering plane:

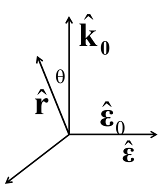
$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left|\hat{\boldsymbol{\varepsilon}}\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$= k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2$$

Assuming both polarizations are equally likely, average cross section is given by:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{k^4 a^6}{2} \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left(\cos^2\theta + 1\right)$$

Scattering by dielectric sphere with permittivity ε and radius *a*:



$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right) = \frac{k^4 a^6}{2} \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left(\cos^2\theta + 1\right)$$