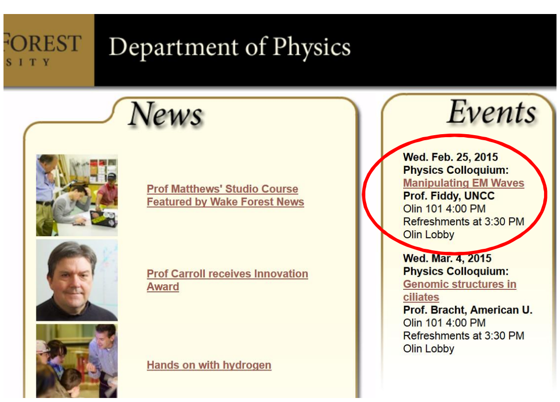
PHY 712 Electrodynamics
9-9:50 AM Olin 103

Plan for Lecture 18:

Complete reading of Chapter 7

1. Summary of complex response functions in electromagnetic fields
2. Summary of TEM plane wave solutions to Maxwell's equations
3. Comments on some homework problems

8	Mon: 02/02/2015	Chap. 4	Multipole analysis	#8	02/04/2015
9	Wed: 02/04/2015	Chap. 4	Dipoles and dielectrics	#9	02/06/2015
10	Fri: 02/06/2015	Chap. 4	Dipoles and dielectrics	#10	02/09/2015
11	Mon: 02/09/2015	Chap. 5	Magnetostatics	#11	02/11/2015
12	Wed: 02/11/2015	Chap. 5	Magnetostatics	#12	02/13/2015
13	Fri: 02/13/2015	Chap. 5	Magnetostatics	#13	02/16/2015
14	Mon: 02/16/2015	Chap. 6	Maxwell's equations	#14	02/18/2015
15	Wed: 02/18/2015	Chap. 6	Electromagnetic energy and force	#15	02/20/2015
16	Fri: 02/20/2015	Chap. 7	Electromagnetic plane waves	#16	02/23/2015
17	Mon: 02/23/2015	Chap. 7	Dielectric media	#17	02/25/2015
18	Wed: 02/25/2015	Chap. 7	Complex dielectrics	#18	02/27/2015
19	Fri: 02/27/2015	Chap. 1-7	Review -- Take home exam distributed		
	Mon. 03/02/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Mon. 03/04/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Wed. 03/06/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Fri. 03/09/2015	*Spring Break*			
	Wed. 03/11/2015	*Spring Break*			
	Fri. 03/13/2015	*Spring Break*			
20	Fri: 03/16/2015	Chap. 8	Wave guides		



WFU Physics Colloquium

TITLE: Manipulating Electromagnetic Waves with Engineered Materials

SPEAKER: Dr. Mike Fiddy,

Optoelectronics Center
University of North Carolina, Charlotte

TIME: Wednesday February 25, 2015 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

Engineered materials or metamaterials offer the promise of extreme refractive index properties (e.g. very large, zero or negative values) that do not arise in nature. The field has attracted a lot of attention because of promised of superresolved imaging and cloaking. One physical mechanism that is exploited to achieve these properties relies on the combined effect of many subwavelength-sized (high Q) circuits or meta-atoms operating close to resonance. Extracting meaningful constitutive parameters like epsilon, mu or

Review: Drude model dielectric function:

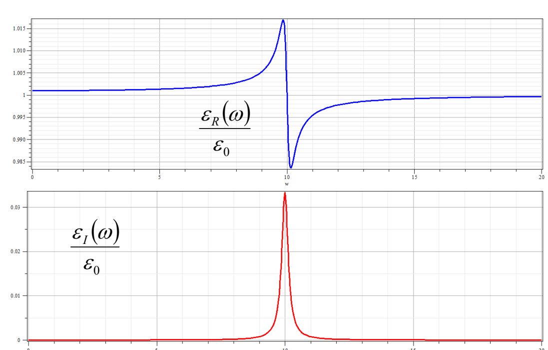
$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{\omega_i^2 - \omega^2}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^2}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{\omega\gamma_i}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^2}$$

Drude model dielectric function:



Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$\text{For } \omega >> \omega_i \quad \frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\right)$$

$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

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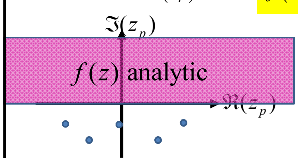
Analysis for Drude model dielectric function – continued --
Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P{}^2 - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P) \leq 0 \Rightarrow f(z)$ is analytic for $\Im(z_P) > 0$



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Because of these analytic properties, Cauchy's integral theorem results in:

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1\right) \frac{1}{\omega' - \omega}$$

$$\text{with} \quad \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

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Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t)=\varepsilon_0\left\{\mathbf{E}(\mathbf{r},t)+\int_0^\infty d\tau\, G(\tau)\mathbf{E}(\mathbf{r},t-\tau)\right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0}-1=\int_0^\infty d\tau\, G(\tau)e^{i\omega\tau}$$

For $\frac{\varepsilon(\omega)}{\varepsilon_0}-1=\frac{N}{\varepsilon_0}\sum_i f_i\frac{q_i^2}{m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$

$$G(\tau)=\frac{N}{\varepsilon_0}\sum_i f_i\frac{q_i^2}{m_i}e^{-\gamma_i\tau/2}\frac{\sin(\nu_i\tau)}{\nu_i}\Theta(\tau)$$

where $\nu_i\equiv\sqrt{\omega_i^2-\gamma_i^2/4}$

Review: Electromagnetic plane waves in isotropic medium with real permeability and permittivity: μ ε.

$$\mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0e^{i\frac{\omega}{c}(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct)}\right)\qquad n^2=c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t)=\frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)=\sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

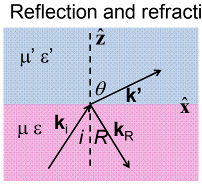
Poynting vector for plane electromagnetic waves:

$$\langle\mathbf{S}\rangle_{avg}=\frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}}=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg}=\frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

Review:
Reflection and refraction between two isotropic media

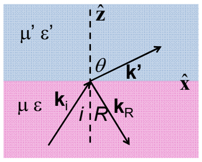


Reflectance, transmittance:

$$\mathrm{R}=\frac{\mathbf{S}_R\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E_{0R}}{E_{0i}}\right|^2\qquad \mathrm{T}=\frac{\mathbf{S}'\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E'_0}{E_{0i}}\right|^2\frac{n'}{n}\frac{\mu}{\mu'}\frac{\cos\theta}{\cos i}$$

Note that $\mathrm{R}+\mathrm{T}=1$

Reflection and refraction between two isotropic media -- continued



For each wave:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}(n\hat{\mathbf{k}}\cdot\mathbf{r} - ct)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Matching condition at interface:

$$n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$$

If $n > n'$, for $i > i_0 \equiv \sin^{-1}\left(\frac{n'}{n}\right)$,

refracted field no longer propagates in medium $\mu'\varepsilon'$

Total internal reflection:

$$n'\cos\theta = i\sqrt{n^2\sin^2 i - n'^2} = i\,n\sqrt{\frac{\sin^2 i}{\sin^2 i_0} - 1}$$

$$\mathbf{E}'(\mathbf{r},t) = e^{-\left(\frac{n\omega}{c}\sqrt{\frac{\sin^2 i}{\sin^2 i_0}-1}\right)z}\,\Re\left(\mathbf{E}'_0 e^{i\frac{\omega}{c}(n\hat{\mathbf{k}}_{\parallel}\cdot\mathbf{r} - ct)}\right)$$

For s-polarization

$$\frac{E_{0R}}{E_{0i}} = \frac{n\cos i - \frac{\mu}{\mu'}n'\cos\theta}{n\cos i + \frac{\mu}{\mu'}n'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}n'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

For p-polarization

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'}n'\cos i - n\cos\theta}{\frac{\mu}{\mu'}n'\cos i + n\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{\frac{\mu}{\mu'}n'\cos i + n\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

Special case: normal incidence (i=0, θ=0)

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n} \qquad \frac{E'_0}{E_{0i}} = \frac{2n}{\frac{\mu}{\mu'}n' + n}$$

Reflectance, transmittance:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2$$

$$\mathrm{T} = \left|\frac{E'_0}{E_{0i}}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'} = \left|\frac{2n}{\frac{\mu}{\mu'}n' + n}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'}$$

Extension to complex refractive index $n = n_R + i\, n_I$

Suppose $\mu = \mu'$, $n = \text{real}$, $n' = n'_R + i n'_I$

Reflectance at normal incidence:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2 = \frac{(n'_R - n)^2 + (n'_I)^2}{(n'_R + n)^2 + (n'_I)^2}$$

Note that for $n'_I >> |n'_R \pm n|$:

$$\mathrm{R} \approx 1$$

Fields near the surface on an ideal conductor

Suppose for an isotropic medium: $\mathbf{D} = \varepsilon_b \mathbf{E}$ $\quad \mathbf{J} = \sigma \mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_b \frac{\partial \mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\varepsilon_b \frac{\partial^2}{\partial t^2}\right)\mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \text{where } \mathbf{k} = (n_R + i n_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i n_R(\omega/c)\hat{\mathbf{k}}\cdot\mathbf{r} - i\omega t}\right)$$

Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c} n_R = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_b}\right)^2} + 1\right)^{1/2}$$

$$\frac{\omega}{c} n_I = \omega\sqrt{\frac{\mu\varepsilon_b}{2}}\left(\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon_b}\right)^2} - 1\right)^{1/2}$$

For $\frac{\sigma}{\omega} >> 1 \qquad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t)$$

Fields near the surface on an ideal conductor -- continued

$$\text{For } \frac{\sigma}{\omega} >> 1 \qquad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1}{\delta}$$

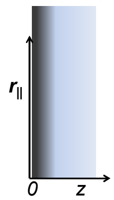
$$\text{In this limit, } \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = c\sqrt{\mu\varepsilon} = n_R + i n_I = \frac{c}{\omega}\frac{1}{\delta}(1+i)$$

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{D}(\mathbf{r},t) = \varepsilon\mathbf{E}(\mathbf{r},t) = \frac{i\sigma}{\omega}\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t) = \mu\mathbf{H}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$



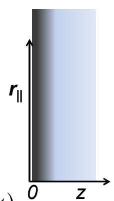
Fields near the surface on an ideal conductor -- continued

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{D}(\mathbf{r},t) = \varepsilon\mathbf{E}(\mathbf{r},t) = \frac{i\sigma}{\omega}\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t) = \mu\mathbf{H}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$



Note that the $\mathbf{H}$ field is larger than $\mathbf{E}$ field so we can write:

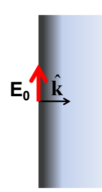
$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta\mu\omega\frac{1-i}{2}\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t)$$

Boundary values for ideal conductor

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \Re\left(\frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)\right)$$



At the boundary of an ideal conductor, the **E** and **H** fields decay in the direction normal to the interface, the field directions are in the plane of the interface.

Waveguide terminology

- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

TEM waves

Transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)

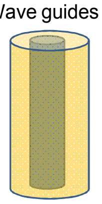
In the free space or within a non-conducting medium; the "normal" electromagnetic modes are TEM:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct)}\right) \qquad n^2 = c^2\mu\varepsilon$$

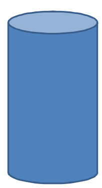
$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

$$\hat{\mathbf{k}}\cdot\mathbf{E} = 0 = \hat{\mathbf{k}}\cdot\mathbf{B}$$

Wave guides

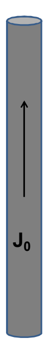


Coaxial cable
TEM modes



Simple optical pipe
TE or TM modes

Comment on HW #11

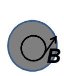


1. Consider an infinitely long wire with radius a, oriented along the **z** axis. There is a steady uniform current inside the wire. Specifically the current is along the z-axis with the magnitude of J_0 for $\rho \le a$ and zero for $\rho > a$, where ρ denotes the radial parameter of the natural cylindrical coordinates of the system.
   a. Find the vector potential (**A**) for all ρ.
   b. Find the magnetic flux field (**B**) for all ρ.

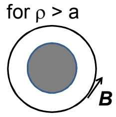
Solution to problem using PHY 114 ideas

In this case, it is convenient to solve part b first.

Top view for $\rho < a$



Top view for $\rho > a$

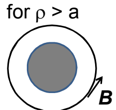


Comment on HW #11 -- continued

Top view for ρ < a

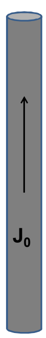
$$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 \int \mathbf{J}\cdot d\mathbf{A}$$
$$2\pi\rho B = \mu_0 J_0 \pi \rho^2$$
$$B = \frac{\mu_0 J_0 \rho}{2}$$
$$\mathbf{B} = \frac{\mu_0 J_0 \rho}{2}\hat{\boldsymbol{\varphi}} = \nabla\times\mathbf{A}$$
$$\mathbf{A} = -\frac{\mu_0 J_0\left(\rho^2 - a^2\right)}{4}\hat{\mathbf{z}}$$

Top view for ρ > a



$$\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 \int \mathbf{J}\cdot d\mathbf{A}$$
$$2\pi\rho B = \mu_0 J_0 \pi a^2$$
$$B = \frac{\mu_0 J_0 a^2}{2\rho}$$
$$\mathbf{B} = \frac{\mu_0 J_0 a^2}{2\rho}\hat{\boldsymbol{\varphi}} = \nabla\times\mathbf{A}$$
$$\mathbf{A} = -\frac{\mu_0 J_0 a^2 \ln\left(\rho/a\right)}{2}\hat{\mathbf{z}}$$

Comment on HW #11 -- continued



Alternative treatment using differential equations:

$$-\nabla^2\mathbf{A} = \begin{cases} \mu_0 J_0 \hat{\mathbf{z}} & \text{for } \rho \le a \\ 0 & \text{for } \rho > a \end{cases}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A_z(\rho)}{\partial\rho} = \begin{cases} \mu_0 J_0 & \text{for } \rho \le a \\ 0 & \text{for } \rho > a \end{cases}$$

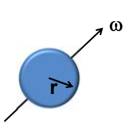
$$A_z(\rho) = \begin{cases} -\frac{\mu_0 J_0 \rho^2}{4} + C_1 & \text{for } \rho \le a \\ C_2 + C_3 \ln(\rho) & \text{for } \rho > a \end{cases}$$

Choosing constants from continuity requirements:

$$A_z(\rho) = \begin{cases} -\frac{\mu_0 J_0 \rho^2}{4} + \frac{\mu_0 J_0 a^2}{4} & \text{for } \rho \le a \\ -\frac{\mu_0 J_0 a^2}{2}\ln(\rho/a) & \text{for } \rho > a \end{cases}$$

$$\mathbf{B} = -\frac{\partial A_z(\rho)}{\partial\rho}\hat{\boldsymbol{\varphi}}$$

Comment on HW #12



A sphere of radius a carries a uniform surface charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential **A** and magnetic field **B** both inside and outside the sphere.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \sigma\ \delta(r'-a)\ \boldsymbol{\omega}\times\mathbf{r}' & \text{for } r' \le a \\ 0 & \text{otherwise} \end{cases}$$

Note that: $\dfrac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm}\dfrac{4\pi}{2l+1}\dfrac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$

and: $\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')\,\mathbf{r}' = \dfrac{r'}{r}\mathbf{r}\,\delta_{l1}.$

Comment on HW #12 -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0 \sigma}{4\pi}\frac{\boldsymbol{\omega}\times\mathbf{r}}{r}\frac{4\pi}{3}\int_0^a r'^3 dr' \, \delta(r'-a)\frac{r_<}{r_>^2}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \sigma}{3}\boldsymbol{\omega}\times\mathbf{r}\begin{cases} a & \text{for } r \le a \\ \dfrac{a^4}{r^3} & \text{for } r > a \end{cases}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 \sigma}{3}\begin{cases} 2\boldsymbol{\omega} a & \text{for } r \le a \\ \dfrac{a^4}{r^3}\left(3\left(\hat{\mathbf{r}}\cdot\boldsymbol{\omega}\right)\hat{\mathbf{r}} - \boldsymbol{\omega}\right) & \text{for } r > a \end{cases}$$