PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 13:

Continue reading Chapter 5

1. **Hyperfine interaction**
2. **Macroscopic magnetization density M**
3. **H field and its relation to B**
4. **Magnetic boundary values**

8	Mon: 02/02/2015	Chap. 4	Multipole analysis	#8	02/04/2015
9	Wed: 02/04/2015	Chap. 4	Dipoles and dielectrics	#9	02/06/2015
10	Fri: 02/06/2015	Chap. 4	Dipoles and dielectrics	#10	02/09/2015
11	Mon: 02/09/2015	Chap. 5	Magnetostatics	#11	02/11/2015
12	Wed: 02/11/2015	Chap. 5	Magnetostatics	#12	02/13/2015
13	Fri: 02/13/2015	Chap. 5	Magnetostatics	#13	02/16/2015
14	Mon: 02/16/2015	Chap. 6	Maxwell's equations	#13	02/18/2015
15	Wed: 02/18/2015	Chap. 6	Electromagnetic energy and force	#15	02/20/2015
16	Fri: 02/20/2015	Chap. 7	Electromagnetic plane waves	#16	02/23/2015
17	Mon: 02/23/2015	Chap. 7	Dielectric media	#17	02/25/2015
18	Wed: 02/25/2015	Chap. 7	Complex dielectrics	#18	02/27/2015
19	Fri: 02/27/2015	Chap. 8	Wave guides		
	Mon. 03/02/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Mon. 03/04/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Mon. 03/06/2015	*APS Meeting*	Take-home exam (no class meeting)		
	Mon. 03/09/2015	*Spring Break*			
	Wed. 03/11/2015	*Spring Break*			
	Fri. 03/13/2015	*Spring Break*			
20	Fri: 02/27/2015	Chap. 8	Wave guides		

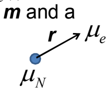
Hyperfine interaction energy:

Interactions between magnetic dipoles

Sources of magnetic dipoles and other sources of magnetism in an atom:

- Intrinsic magnetic moment of a nucleus μ_N
- Intrinsic magnetic moment of an electron μ_e
- Magnetic field due to electron current $\mathbf{J}_e(\mathbf{r})$

Interaction energy between a magnetic dipole ***m*** and a magnetic field **B:** $E_{int} = -\mathbf{m}\cdot\mathbf{B}$



In this case: $E_{int} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J}_e}(0)$

$$\mathbf{B}_{\boldsymbol{\mu}_N}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{3\hat{\mathbf{r}}(\boldsymbol{\mu}_N\cdot\hat{\mathbf{r}})-\boldsymbol{\mu}_N}{r^3}+\frac{8\pi}{3}\boldsymbol{\mu}_N\delta^3(\mathbf{r})\right\}$$

Hyperfine interaction energy: -- continued

$$E_{int} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J_e}}(0)$$

Evaluation of the magnetic field at the nucleus due to the electron current density:
The vector potential associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction $\psi_{nlm_l}(\mathbf{r})$ can be written:

$$\mathbf{A}_{\mathbf{J}_e}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'}$$

We want to evaluate the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ in the vicinity of the nucleus $(\mathbf{r} \to 0)$.

Hyperfine interaction energy: -- continued

$$\mathbf{B}_{\mathbf{J}_e}(0) = \nabla \times \mathbf{A}_{\mathbf{J}_e}\Big|_{\mathbf{r}\to 0} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \nabla \times \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{(\mathbf{r}-\mathbf{r}') \times (\hat{\mathbf{z}} \times \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'}\Bigg|_{\mathbf{r}\to 0}$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\mathbf{r}' \times (\hat{\mathbf{z}} \times \mathbf{r}')}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'}$$

$$\hat{\mathbf{r}}' \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}}') = \hat{\mathbf{z}}(1 - \cos^2\theta') - \hat{\mathbf{x}}\cos\theta' \sin\theta' \cos\phi' - \hat{\mathbf{y}}\cos\theta' \sin\theta' \sin\phi').$$

$$\mathbf{B}_{\mathbf{o}}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}} r'^2 \sin^2\theta'}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'} = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \hat{\mathbf{z}} \int d^3r' \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^3}$$

$$= -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \hat{\mathbf{z}} \left\langle \frac{1}{r'^3} \right\rangle$$

Hyperfine interaction energy: -- continued

$$E_{int} \equiv H_{HF} = -\mu_e \cdot \mathbf{B}_{\mu_N} - \mu_N \cdot \mathbf{B}_{\mathbf{J_e}}(0)$$

Putting all of the terms together:

$$H_{\mathrm{HF}} = -\frac{\mu_0}{4\pi}\left(\left\langle \frac{3(\boldsymbol{\mu}_\mathbf{N} \cdot \hat{\mathbf{r}})(\boldsymbol{\mu}_\mathbf{e} \cdot \hat{\mathbf{r}}) - \boldsymbol{\mu}_\mathbf{N} \cdot \boldsymbol{\mu}_\mathbf{e}}{r^3} + \frac{8\pi}{3}\boldsymbol{\mu}_\mathbf{N} \cdot \boldsymbol{\mu}_\mathbf{e}\delta^3(\mathbf{r})\right\rangle + \frac{e}{m_e}\left\langle \frac{\mathbf{L} \cdot \boldsymbol{\mu}_\mathbf{N}}{r^3}\right\rangle\right).$$

In this expression the brackets $\langle\ \rangle$ indicate evaluating the expectation value relative to the electronic state.

Macroscopic dipolar effects --
Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2}\int d^3r\, \mathbf{r}\times\mathbf{J}(\mathbf{r})$$

Note that the intrinsic spin of elementary particles is associated with a magnetic dipole moment, but we often do not have a detailed knowledge of **J**(**r**).

Vector potential for magnetic dipole moment

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{|\mathbf{r}|^3}$$

Macroscopic magnetization

$$\mathbf{M}(\mathbf{r}) = \sum_i \mathbf{m}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$

Vector potential due to "free" current $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ and macroscopic magnetization **M**(**r**). Note: the designation $\mathbf{J}_{\mathbf{free}}(\mathbf{r})$ implies that this current does not also contribute to the magnetization density.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \left(\frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right)$$

Note that :

$$\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{M}(\mathbf{r}')\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

$$= -\nabla'\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) + \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that for the case that $\nabla \cdot \mathbf{A} = 0$:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times (\nabla \times \mathbf{A}(\mathbf{r})) = -\nabla^2 \mathbf{A}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \left(4\pi\delta^3(\mathbf{r}-\mathbf{r}')\right)\left(\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')\right)$$

$$= \mu_0\left(\mathbf{J}_{free}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r})\right)$$

$$\Rightarrow \nabla \times \left(\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Magnetic field contributions

$$\nabla \times \left(\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Define the magnetic flux density:

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\nabla \times \left(\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})\right) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Note that $\mathbf{B}(\mathbf{r}) \equiv$ the magnetic flux density

Define $\mathbf{H}(\mathbf{r}) \equiv$ the magnetic field

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

Summary of equations of magnetostatics:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

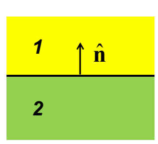
$$\mathbf{B}(\mathbf{r}) = \mu_0\left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r}) = 0$:

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

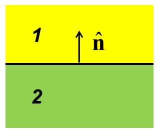


For the case that $\mathbf{J}_{free}(\mathbf{r})$:

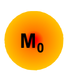
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

At boundary:

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$
$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



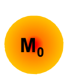
Example magnetostatic boundary value problem



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0 \qquad \Rightarrow \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$$
$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 = \mu_0 \nabla \cdot (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$
$$\Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

$$\nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \int d^3r' \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= -\frac{1}{4\pi} \int d^3r' \left[\nabla' \cdot \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \mathbf{M}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right]$$

$$= -\frac{1}{4\pi} \nabla \cdot \int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Example magnetostatic boundary value problem -- continued

M_0

$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases} \quad \Phi_H(\mathbf{r}) = -\frac{1}{4\pi}\nabla\cdot\int d^3r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

For this example:

$$\Phi_H(\mathbf{r}) = -\frac{M_0}{4\pi}\frac{\partial}{\partial z}\left(4\pi\int_0^a r'^2\, dr' \frac{1}{r_>}\right)$$

For $r \le a$: $\Phi_H(\mathbf{r}) = -M_0\frac{\partial}{\partial z}\left(\frac{a^2}{2} - \frac{r^2}{6}\right) = \frac{M_0 z}{3}$

For $r > a$: $\Phi_H(\mathbf{r}) = -M_0\frac{\partial}{\partial z}\left(\frac{a^3}{3r}\right) = \frac{M_0 a^3 z}{3r^3}$

Example magnetostatic boundary value problem -- continued

M_0

$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0\hat{\mathbf{z}} & r \le a \\ 0 & r > a \end{cases}$$

For $r \le a$: $\Phi_H(\mathbf{r}) = \frac{M_0 z}{3}$ $\quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\frac{M_0}{3}\hat{\mathbf{z}}$

For $r > a$: $\Phi_H(\mathbf{r}) = \frac{M_0 a^3 z}{3r^3}$ $\quad \mathbf{H}(\mathbf{r}) = -\nabla\Phi_H(\mathbf{r}) = -\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

$\mathbf{B}(\mathbf{r}) = \mu_0\left(\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})\right)$

For $r \le a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0\hat{\mathbf{z}}}{3}$ $\quad \mathbf{B}(\mathbf{r}) = \mu_0\frac{2M_0\hat{\mathbf{z}}}{3}$

For $r > a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(\mathbf{r}) = -\mu_0\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$$

Check boundary values:

For $r \le a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0\hat{\mathbf{z}}}{3}$ $\quad \mathbf{H}(a\hat{\mathbf{r}})\times\hat{\mathbf{r}} = -\frac{M_0}{3}\hat{\mathbf{z}}\times\hat{\mathbf{r}}$

For $r > a$: $\mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

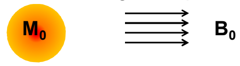
$$\mathbf{H}(a\hat{\mathbf{r}})\times\hat{\mathbf{r}} = -\frac{M_0 a^3}{3}\frac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{a^3}$$

For $r \le a$: $\mathbf{B}(\mathbf{r}) = \mu_0\frac{2M_0\hat{\mathbf{z}}}{3}$ $\quad \mathbf{B}(a\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} = \mu_0\frac{2M_0}{3}\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}$

For $r > a$: $\mathbf{B}(\mathbf{r}) = -\mu_0\frac{M_0 a^3}{3}\left(\frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5}\right)$

$$\mathbf{B}(a\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} = -\mu_0\frac{M_0 a^3}{3}\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}\left(\frac{1}{a^3} - \frac{3a^2}{a^5}\right)$$

Variation; magnetic sphere plus external field $\mathbf{B}_0$



$$\mathbf{M}(\mathbf{r}) = \begin{cases} \mathbf{M}_0 & r \leq a \\ 0 & r > a \end{cases}$$

By superposition:

For $r \leq a$:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \mu_0 \frac{2}{3}\mathbf{M}_0$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0}\mathbf{B}_0 - \frac{1}{3}\mathbf{M}_0$$

$$\mathbf{B}(\mathbf{r}) + 2\mu_0 \mathbf{H}(\mathbf{r}) = 3\mathbf{B}_0$$

For an isotropic "paramagnetic" material, $\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$

$$\mathbf{M}_0 = \frac{3}{\mu_0}\left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right)\mathbf{B}_0$$

Summary of equations of magnetostatics:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that $\mathbf{J}_{free}(\mathbf{r})$:

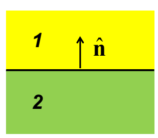
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

At boundary:

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



Magnetism in materials

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

For materials with linear magnetism:

$$\mathbf{B} = \mu \mathbf{H}$$

$\mu > \mu_0 \Rightarrow$ paramagnetic material

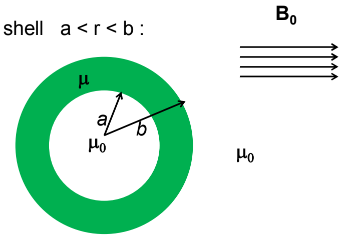
$\mu < \mu_0 \Rightarrow$ diamagnetic material

For ferromagnetic, antiferromagnetic materials

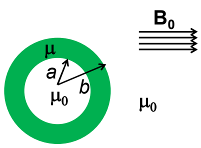
$\mathbf{B} = f(\mathbf{H})$ (with hysteresis)

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$

Spherical shell a < r < b :



Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



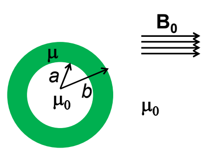
For this case :

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$
$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$
$$\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$$

Continuity at boundaries :

$$\mathbf{H} \times \hat{\mathbf{n}} = \text{continuous}$$
$$\mathbf{B} \cdot \hat{\mathbf{n}} = \text{continuous}$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



Let : $\mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$

$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = 0$

For $0 \le r \le a$ $\qquad \Phi_H(\mathbf{r}) = \sum_l \delta_l r^l P_l(\cos\theta)$

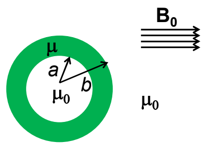
For $a \le r \le b$ $\qquad \Phi_H(\mathbf{r}) = \sum_l \left(\beta_l r^l + \frac{\gamma_l}{r^{l+1}} \right) P_l(\cos\theta)$

For $r \ge b$ $\qquad \Phi_H(\mathbf{r}) = -\frac{B_0}{\mu_0} r\cos\theta + \sum_l \frac{\alpha_l}{r^{l+1}} P_l(\cos\theta)$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued

Applying boundary conditions

(only $l=1$ terms contribute):



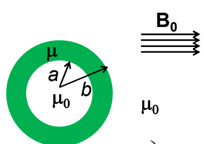
At $r=a$ $\quad \delta_1 = \frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{a^3}\right)$

$$a\delta_1 = a\beta_1 + \frac{\gamma_1}{a^2}$$

At $r=b$ $\quad \frac{\mu}{\mu_0}\left(\beta_1 - 2\frac{\gamma_1}{b^3}\right) = -\frac{B_0}{\mu_0} - 2\frac{\alpha_1}{b^3}$

$$b\beta_1 + \frac{\gamma_1}{b^2} = -b\frac{B_0}{\mu_0} + \frac{\alpha_1}{b^2}$$

Example: permalloy, mumetal $\mu/\mu_0 \sim 10^4$ -- continued



When the dust clears:

$$\delta_1 = \left(\frac{-9\mu/\mu_0}{(2\mu/\mu_0+1)(\mu/\mu_0+2) - 2(a/b)^3(\mu/\mu_0-1)^2}\right)\frac{B_0}{\mu_0}$$

$$\approx \frac{1}{\mu/\mu_0}\left(\frac{-9/2}{\left(1-(a/b)^3\right)}\frac{B_0}{\mu_0}\right)$$

Energy associated with magnetic fields

Note: We previously used without proof --

the force on a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is:

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

This implies that energy associated with aligning a magnetic dipole $\mathbf{m}$ in an external $\mathbf{B}$ field is given by:

$$U = -\mathbf{m}\cdot\mathbf{B}$$

Macroscopic energies --

It can be shown that: $\quad W_B = \frac{1}{2}\int d^3r\, \mathbf{B}(\mathbf{r})\cdot\mathbf{H}(\mathbf{r})$

In analogy to: $\quad W_E = \frac{1}{2}\int d^3r\, \mathbf{E}(\mathbf{r})\cdot\mathbf{D}(\mathbf{r})$