PHY 712 Electrodynamics
9-9:50 AM MWF Olin 103

Plan for Lecture 7:

Start reading Chapter 3

Solution of Poisson equation in for special geometries –

A. Cylindrical

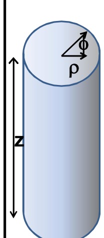
B. Spherical

Course schedule for Spring 2015

(Preliminary schedule -- subject to frequent adjustment.)

	Lecture date	JDJ Reading	Topic	Assign.	Due date
1	Mon: 01/12/2015	Chap. 1	Introduction, units and Poisson equation	#1	01/23/2015
2	Wed: 01/14/2015	Chap. 1	Electrostatic energy calculations	#2	01/23/2015
	Fri: 01/16/2015	No class	NAWH out of town		
	Mon: 01/19/2015	No class	MLK Holiday		
3	Wed: 01/21/2015	Chap. 1	Poisson equation and Green's theorem	#3	01/23/2015
4	Fri: 01/23/2015	Chap. 1 & 2	Green's functions in Cartesian coordinates	#4	01/26/2015
5	Mon: 01/26/2015	Chap. 1 & 2	Brief introduction to grid solution methods	#5	01/28/2015
6	Wed: 01/28/2015	Chap. 2	Method of images	#6	01/30/2015
7	Fri: 01/30/2015	Chap. 3	Cylindrical and spherical geometries	#7	02/02/2015
8	Mon: 02/02/2015	Chap. 4	Multipole analysis	#8	02/04/2015

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence

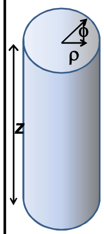


Laplace equation : $\nabla^2 \Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \phi, z) = R(\rho) Q(\phi) Z(z)$$

Cylindrical geometry continued:



Laplace equation : $\nabla^2\Phi = 0$

$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$

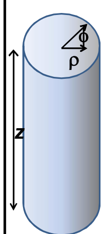
One possibility :

$$\frac{d^2Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry continued:



Laplace equation : $\nabla^2\Phi = 0$

$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$

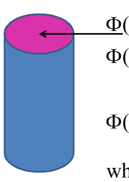
Another possibility :

$$\frac{d^2Z}{dz^2} + k^2 Z = 0 \qquad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right)R = 0 \qquad \Rightarrow I_m(k\rho), K_m(k\rho)$$

Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*

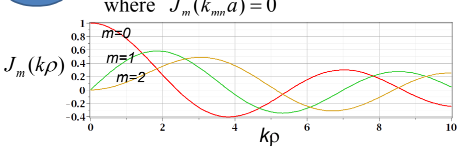


$\Phi(\rho,\phi,z=L) = V(\rho,\phi)$

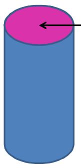
$\Phi(\rho,\phi,z) = 0$ on all other boundaries

$$\Phi(\rho,\phi,z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin(m\phi + \alpha_{mn})$$

where $J_m(k_{mn}a) = 0$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*



$$\Phi(\rho,\varphi,z=L)=V(\rho,\varphi)$$

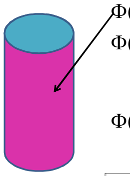
$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\varphi,z)=\sum_{n,m}A_{mn}J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\varphi+\alpha_{mn}\right)$$

If $V(\rho,\varphi)$ is an even function of φ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi}d\varphi\cos\left(m\varphi\right)\int_0^a\rho d\rho J_m(k_{mn}\rho)V(\rho,\varphi)}{\sinh(k_{mn}L)\int_0^{2\pi}d\varphi\cos^2\left(m\varphi\right)\int_0^a\rho d\rho J_m{}^2(k_{mn}\rho)}$$

Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *ρ=a*

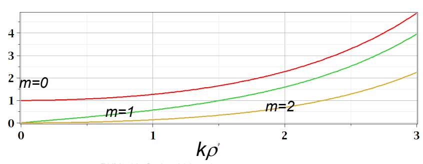


$$\Phi(\rho=a,\phi,z)=V(\phi,z)$$

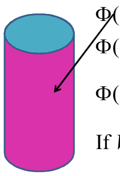
$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m}A_{mn}I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\phi+\alpha_{mn}\right)$$

$I_m(k\rho)$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *ρ=a*



$$\Phi(\rho=a,\varphi,z)=V(\varphi,z)$$

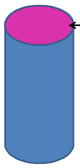
$$\Phi(\rho,\varphi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\varphi,z)=\sum_{n,m}A_{mn}I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\varphi+\alpha_{mn}\right)$$

If $V(z,\varphi)$ is an even function of φ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi}d\varphi\cos\left(m\varphi\right)\int_0^L dz\sin\left(\frac{n\pi z}{L}\right)V(z,\varphi)}{I_m\left(\frac{n\pi a}{L}\right)\int_0^{2\pi}d\varphi\cos^2\left(m\varphi\right)\int_0^L dz\sin^2\left(\frac{n\pi z}{L}\right)}$$

Green's function for Dirchelet boundary value inside cylindar:



$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho=a,\phi,z)=0,\ \Phi(\rho,\phi,z=0)=0$$

Expansion in terms of Bessel function zeros: $J_m(k_{mn}a)=0$

$$G(\rho,\rho',\phi,\phi',z,z')=$$

$$\frac{8\pi}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}J_m(k_{mn}\rho)J_m(k_{mn}\rho')\sinh(k_{mn}z_<)\sinh(k_{mn}(L-z_>))}{k_{mn}(J_{m+1}(k_{mn}a))^2\sinh(k_{mn}L)}$$

$$\Phi(\rho,\phi,z)=\frac{1}{4\pi\varepsilon_0}\int_V d\phi'\rho'd\rho'dz'G(\rho,\rho',\phi,\phi',z,z')\rho(\rho',\phi',z')$$

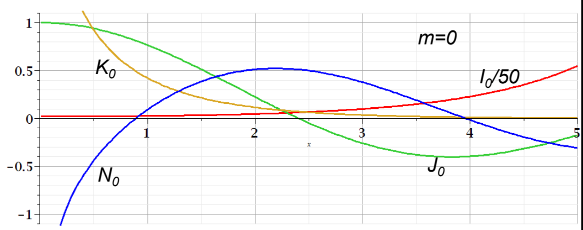
$$+\frac{1}{4\pi}\int_{S;z'=L} d\phi'\rho'd\rho'\left.\frac{\partial G(\rho,\rho',\phi,\phi',z,z')}{\partial z'}\right|_{z'=L}V(\rho',\phi')$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$



Comments on cylindrical Bessel functions

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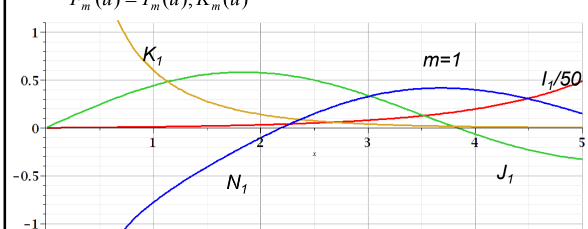


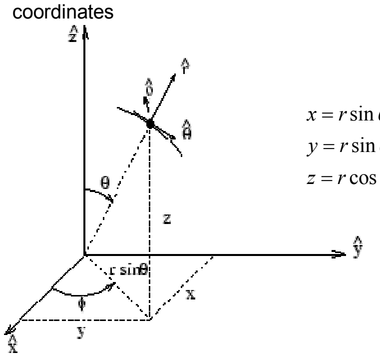
Some useful identities involving cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(1-\frac{m^2}{u^2}\right)\right)J_m(u)=0 \quad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: x_{mn}; $\quad J_m(x_{mn})=0$

$$\int_0^a \rho d\rho J_m(x_{mn}\rho/a)J_m(x_{mn'}\rho/a)=\frac{a^2}{2}\left(J_{m+1}(x_{mn})\right)^2\delta_{nn'}$$

Poisson and Laplace equation in spherical polar coordinates



$$x=r\sin\theta\cos\varphi$$
$$y=r\sin\theta\sin\varphi$$
$$z=r\cos\theta$$

http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html

Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r,\theta,\phi)$:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)\Phi=0$$

$$\Phi(r,\theta,\phi)=\sum_{lm}R_{lm}(r)Y_{lm}(\theta,\phi)$$

Spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)Y_{lm}(\theta,\phi)=-l(l+1)Y_{lm}(\theta,\phi)$$

Properties of spherical harmonic functions

$$Y_{lm}(\theta,\phi) = (-1)^m Y_{l(-m)}^*(\theta,\phi) \quad \text{(standard Condon - Shortley convention)}$$

$$\int d\Omega Y_{lm}(\theta,\varphi) Y_{l'm'}^*(\theta,\varphi) \equiv \int \sin\theta \, d\theta \, d\varphi \, Y_{lm}(\theta,\varphi) Y_{l'm'}^*(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

Completeness:

$$\sum_{lm} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') \equiv \delta(\cos\theta - \cos\theta')\delta(\phi - \phi')$$

Relationship to Legendre polynomials:

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

Useful identity:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r \to \infty$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') \right)$$

Example -- continued

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') \right)$$

Suppose: $\rho(\mathbf{r}') = \frac{Q}{a^3\pi^{3/2}} e^{-r'^2/a^2}$

$$\int d\Omega' \, Y_{lm}^*(\theta',\varphi') = \sqrt{4\pi}\delta_{l0}\delta_{m0}$$

$$\Phi(\mathbf{r}) = \frac{4\pi}{4\pi\varepsilon_0} \int_0^\infty r'^2 \, dr' \frac{r_<^0}{r_>^1} \frac{Q}{a^3\pi^{3/2}} e^{-r'^2/a^2}$$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{\text{erf}(r/a)}{r}$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Elements of "proof":

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r_>\left(1+\left(\frac{r_<}{r_>}\right)^2-2\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right)^{1/2}}=$$

$$=\frac{1}{r_>}\left(1+\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'+\left(\frac{r_<}{r_>}\right)^2\left(\frac{3}{2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2-\frac{1}{2}\right)+\cdots\right)$$

$$=\frac{1}{r_>}\left(\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\right)$$

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Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Elements of "proof" -- continued:

Sum rule for spherical harmonics:

$$P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\hat{\mathbf{r}})\,Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}}=\hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')=1$

$$\Rightarrow\frac{4\pi}{2l+1}\sum_{m=-l}^{l}|Y_{lm}(\hat{\mathbf{r}})|^2=1$$

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Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}})=\frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm1)}(\hat{\mathbf{r}})=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}})=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2(\pm2)}(\hat{\mathbf{r}})=\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm2i\phi}$$

$$Y_{2(\pm1)}(\hat{\mathbf{r}})=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}})=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$$

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