PHY 770 -- Statistical Mechanics
12:00* - 1:45 PM TR Olin 107

Instructor: Natalie Holzwarth (Olin 300)
Course Webpage: http://www.wfu.edu/~natalie/s14phy770

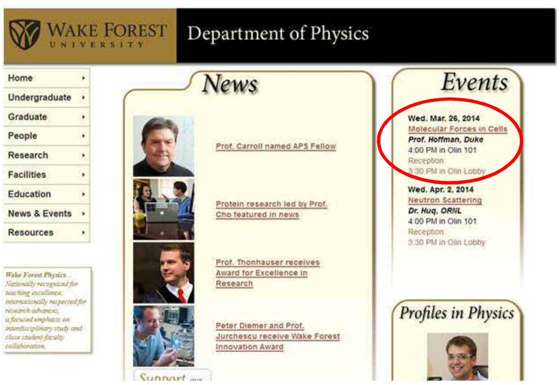
Lecture 16

Chap. 7 – Brownian motion and other non-equilibrium phenomena

- ❑ Overview
- ❑ Langevin equation
- ❑ Correlation function and spectral density
- ❑ Fokker-Planck equation

*Partial make-up lecture -- early start time

2	Thu: 01/16/2014	Chap. 3	Review of macroscopic thermodynamics	#2	02/04/2014
3	Tue: 01/21/2014	Chap. 3	Thermodynamic potentials	#3	02/04/2014
4	Tue: 01/21/2014	Chap. 3	Thermodynamic stability	#4	02/04/2014
5	Thu: 01/23/2014	Chap. 3	Thermodynamic stability	#5	02/04/2014
	Tue: 01/28/2014		NAWH out of town - no class		
	Thu: 01/30/2014		NAWH out of town - no class		
6	Tue: 02/04/2014	Chap. 4	Phase transitions	#6	02/11/2014
7	Thur: 02/06/2014	Chap. 2	Microscopic analysis of entropy	#7	02/11/2014
8	Thur: 02/06/2014	Chap. 2	Microscopic analysis of entropy	#8	02/11/2014
9	Tue: 02/11/2014	Chap. 2	Microscopic analysis of entropy	#9	02/18/2014
	Thur: 02/13/2014		Class cancelled due to weather		
10	Tue: 02/18/2014	Chap. 5	Equilibrium Statistical Mechanics	#10	02/27/2014
11	Tue: 02/18/2014	Chap. 5	Equilibrium Statistical Mechanics	#11	02/27/2014
12	Thur: 02/20/2014	Chap. 5	The Ising model (class 12-1:45 PM)	#12	02/27/2014
	Tue: 02/25/2014		NAWH out of town -- no class		
13	Thur: 02/27/2014	Chap. 6	Grand partition function (class 12-1:45 PM)		
	Tue: 03/04/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Thur: 03/06/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Tue: 03/11/2014		Spring break (no class meeting)		
	Thur: 03/13/2014		Spring break (no class meeting)		
14	Tue: 03/18/2014	Chap. 6	Fermi and Bose particles (class 12-1:45 PM; Exam due)	#13	03/25/2014
15	Thur: 03/20/2014	Chap. 6	Interacting particles (class 12-1:45 PM)	#14	03/25/2014
16	Tue: 03/25/2014	Chap. 7	Langevin equation (class 12-1:45 PM)	#15	04/01/2014
17	Thur: 03/27/2014	Chap. 7	Fokker-Planck equation (class 12-1:45 PM)	#16	04/01/2014



WFU Physics Colloquium

TITLE: Measuring Molecular Forces Across Specific Proteins in Living Cells

SPEAKER: Professor Brent D. Hoffman,

Department of Biomedical Engineering, Duke University

TIME: Wednesday March 26, 2014 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

In vivo, cells adhere to the deformable extracellular matrix (ECM) that is both a source of applied forces and a means of mechanical support. Cells detect and interpret mechanical signals, such as force and rigidity, from the ECM through mechanotransduction. While the connections between cells and the ECM, mediated by structures called focal adhesions (FAs), are primary determinants of mechanotransduction, the molecular mechanisms mediating this process are largely unknown. Progress has been limited by an inability to measure dynamic forces across proteins in living cells. Therefore we developed an experimentally calibrated Forster resonance energy transfer (FRET)-based biosensor that measures forces across specific proteins with pico-Newton sensitivity. The sensor has been applied to vinculin, a critical linker protein in the connections between the integrins and

3/25/2014 PHY 770 Spring 2014 -- Lecture 16 4

http://famousbiologists.org/robert-brown/

Robert Brown

Born: Dec 21, 1773 in Montrose, Scotland

Died: Jun 10, 1858 (at age 84) in London, England

Nationality: Scottish

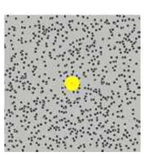
Famous For: Brownian motion

Robert Brown was a botanist from Scotland who was a pioneer in microscopy. He was among the first botanists to describe the nucleus of cells while he also discovered Brownian motion. He was also highly influential in paleobotany, the study of prehistoric plant life. Brown traveled extensively in Australia with navigator Matthew Flinders, making many discoveries about Australian plants.

3/25/2014 PHY 770 Spring 2014 -- Lecture 16 5

Brownian motion:

Phonomenon: Under a microscope a large particle (~1 μ in diameter) immersed in a fluid with the same density as the particle, appears to be in a state of agitation, undergoing rapid and random motions.



http://upload.wikimedia.org/wikipedia/commons/c/c2/Brownian_motion_large.gif

3/25/2014 PHY 770 Spring 2014 -- Lecture 16 6

Brownian motion:

Description based on the Langevin equation of motion

Consider a particle of mass m and radius a immersed in a fluid of particles (of mass much smaller than m) undergoing Brownian motion. The fluid gives rise to a retarding force (friction) that is proportional to the velocity $v(t)$ and a random force $\xi(t)$ due to the random density fluctuations in the fluid.

friction coefficient

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m} v(t) + \frac{1}{m}\xi(t)$$

Brownian motion and Langevin equation of motion -- continued

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m} v(t) + \frac{1}{m}\xi(t)$$

Properties of random force:

$$\left\langle \xi(t) \right\rangle_{\xi} = 0$$

$$\left\langle \xi(t_1)\xi(t_2) \right\rangle_{\xi} = g\delta(t_1 - t_2)$$

For initial conditions:

$$x(t) = x_0; \qquad v(t) = v_0$$

$$\Rightarrow v(t) = v_0 e^{-(\gamma/m)t} + \frac{1}{m}\int_0^t ds\ e^{-(\gamma/m)(t-s)}\ \xi(s)$$

$$x(t) = x_0 + \frac{m}{\gamma} v_0 \left(1 - e^{-(\gamma/m)t}\right) + \frac{1}{\gamma}\int_0^t ds\ \left(1 - e^{-(\gamma/m)(t-s)}\right)\ \xi(s)$$

Brownian motion and Langevin equation of motion – continued

Note that since *ξ(t)* is a stochastic variable, so is *v(t)* and *x(t)*

$$v(t) = v_0 e^{-(\gamma/m)t} + \frac{1}{m}\int_0^t ds\ e^{-(\gamma/m)(t-s)}\ \xi(s)$$

$$\left\langle v(t_1)v(t_2) \right\rangle_{\xi} = {v_0}^2 e^{-(\gamma/m)(t_1+t_2)} + \frac{g}{m^2}\int_0^{t_2} ds_2 \int_0^{t_1} ds_1\ \delta(s_2 - s_1) e^{-(\gamma/m)(t_1-s_1)} e^{-(\gamma/m)(t_2-s_2)}$$

$$= \left({v_0}^2 - \frac{g}{2m\gamma}\right) e^{-(\gamma/m)(t_1+t_2)} + \frac{g}{2m\gamma} e^{-(\gamma/m)(|t_1-t_2|)}$$

$$\left\langle \left(x(t) - x_0\right)^2 \right\rangle_{\xi} = \frac{m^2}{\gamma^2}\left({v_0}^2 - \frac{g}{2m\gamma}\right)\left(1 - e^{-(\gamma/m)t}\right)^2 + \frac{g}{\gamma^2}\left(t - \frac{m}{\gamma}\left(1 - e^{-(\gamma/m)t}\right)\right)$$

Brownian motion and Langevin equation of motion – continued

Note that since *ξ(t)* is a stochastic variable, so is *v(t)* and *x(t)*

$$\left\langle v(t_1)v(t_2)\right\rangle_\xi == \left(v_0{}^2 - \frac{g}{2m\gamma}\right)e^{-(\gamma/m)(t_1+t_2)} + \frac{g}{2m\gamma}e^{-(\gamma/m)(|t_1-t_2|)}$$

$$\left\langle \left(x(t)-\mathrm{x}_0\right)^2\right\rangle_\xi = \frac{m^2}{\gamma^2}\left(v_0{}^2 - \frac{g}{2m\gamma}\right)\left(1-e^{-(\gamma/m)t}\right)^2 + \frac{g}{\gamma^2}\left(t-\frac{m}{\gamma}\left(1-e^{-(\gamma/m)t}\right)\right)$$

Under conditions of thermal equilibrium, argue that

$$\left\langle v_0{}^2\right\rangle_T = \frac{kT}{m}$$

$$\left\langle v_0{}^2\right\rangle_T - \frac{g}{2m\gamma} = 0 \qquad \Rightarrow g = 2m\gamma\left\langle v_0{}^2\right\rangle_T = 2kT\gamma$$

$$\left\langle\left\langle v(t_1)v(t_2)\right\rangle_\xi\right\rangle_T = \frac{kT}{m}e^{-(\gamma/m)(|t_1-t_2|)}$$

$$\left\langle\left\langle \left(x(t)-\mathrm{x}_0\right)^2\right\rangle_\xi\right\rangle_T = \frac{2kT}{\gamma}\left(t-\frac{m}{\gamma}\left(1-e^{-(\gamma/m)t}\right)\right)$$

Brownian motion and Langevin equation of motion – continued

It is interesting to take the Fourier transform of the correlation function

$$\left\langle\left\langle v(t_1)v(t_2)\right\rangle_\xi\right\rangle_T = \frac{kT}{m}e^{-(\gamma/m)(|t_1-t_2|)}$$

$$S_{v,v}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\left\langle\left\langle v(t_1+\tau)v(t_1)\right\rangle_\xi\right\rangle_T = \frac{kT}{m}\int_{-\infty}^{\infty} d\tau e^{i\omega\tau}e^{-(\gamma/m)|\tau|}$$

$$= \frac{2kT}{m}\frac{\gamma/m}{\omega^2+(\gamma/m)^2}$$

Spectral properties of random force:

$$\left\langle \xi(t)\right\rangle_\xi = 0$$

$$\left\langle \xi(t_1)\xi(t_2)\right\rangle_\xi = g\delta(t_1-t_2)$$

$$S_{\xi,\xi}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\left\langle\left\langle \xi(t_1+\tau)\xi(t_1)\right\rangle_\xi\right\rangle_T = g = 2\gamma kT$$

Example of Brownian system

Consider a particle of mass *m* attached to a harmonic spring with spring constant $m\omega_0{}^2$ constrained to move in one dimension:

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) - \omega_0{}^2 x + \frac{1}{m}\xi(t) \qquad v(t) = \frac{dx(t)}{dt}$$

Assume that the particle is initially in thermal equilibrium

with $\left\langle v_0^2\right\rangle_T = \frac{kT}{m}$ and $\omega_0{}^2\left\langle x_0^2\right\rangle_T = kT$

Solution of equation:

Defining $\Gamma \equiv \frac{\gamma}{m}$ $\qquad \Delta \equiv \sqrt{\Gamma^2-\omega_0^2}$

$$C(t) \equiv \cosh(\Delta t) - \frac{\Gamma}{\Delta}\sinh(\Delta t)$$

$$v(t) = v_0 e^{-\Gamma t}C(t) - \frac{\omega_0^2 x_0}{\Delta}e^{-\Gamma t}\sinh(\Delta t) + \frac{1}{m}\int_0^t dt'\,\xi(t')e^{-\Gamma(t-t')}C(t-t')$$

Example of Brownian system -- continued

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) - \omega_0{}^2 x + \frac{1}{m}\xi(t) \qquad v(t) = \frac{dx(t)}{dt}$$

$$v(t) = v_0 e^{-\Gamma t}C(t) - \frac{\omega_0^2 x_0}{\Delta}e^{-\Gamma t}\sinh(\Delta t) + \frac{1}{m}\int_0^t dt'\,\xi(t')e^{-\Gamma(t-t')}C(t-t')$$

Computing thermally averaged correlation function, noting that

$\langle v_0 x_0\rangle_T = 0$ assuming $t_2 > t_1$

$$\left\langle\langle v(t_2)v(t_1)\rangle_\xi\right\rangle_T = \langle v_0{}^2\rangle_T\, e^{-\Gamma(t_2+t_1)}C(t_2)C(t_1)$$

$$-\frac{\omega_0^4\langle x_0{}^2\rangle_T}{\Delta^2}e^{-\Gamma(t_2+t_1)}\sinh(\Delta t_1)\sinh(\Delta t_2)$$

$$+\frac{g}{m^2}\int_0^{t_1} dt'\,e^{-\Gamma(t_1-t')}e^{-\Gamma(t_2-t')}C(t_1-t')C(t_2-t')$$

3/25/2014 PHY 770 Spring 2014 -- Lecture 16 13

Example of Brownian system -- continued

Notation: $\Gamma \equiv \frac{\gamma}{m} \qquad \Delta \equiv \sqrt{\Gamma^2 - \omega_0^2}$

For: $t_2 > t_1$

$$\left\langle\langle v(t_2)v(t_1)\rangle_\xi\right\rangle_T = \langle v_0{}^2\rangle_T\, e^{-\Gamma(t_2+t_1)}C(t_2)C(t_1)$$

$$-\frac{\omega_0^4\langle x_0{}^2\rangle_T}{\Delta^2}e^{-\Gamma(t_2+t_1)}\sinh(\Delta t_1)\sinh(\Delta t_2)$$

$$+\frac{g}{m^2}\int_0^{t_1} dt'\,e^{-\Gamma(t_1-t')}e^{-\Gamma(t_2-t')}C(t_1-t')C(t_2-t')$$

Assuming that $g = 4\gamma kT$; after some algebra:

$$\left\langle\langle v(t_1+\tau)v(t_1)\rangle_\xi\right\rangle_T = \frac{kT}{m}e^{-\Gamma|\tau|}\left(\cosh(\Delta|\tau|) - \frac{\Gamma}{\Delta}\sinh(\Delta|\tau|)\right)$$

3/25/2014 PHY 770 Spring 2014 -- Lecture 16 14

Example of Brownian system -- continued

Notation: $\Gamma \equiv \frac{\gamma}{m} \qquad \Delta \equiv \sqrt{\Gamma^2 - \omega_0^2}$

$$\left\langle\langle v(t_1+\tau)v(t_1)\rangle_\xi\right\rangle_T = \frac{kT}{m}e^{-\Gamma|\tau|}\left(\cosh(\Delta|\tau|) - \frac{\Gamma}{\Delta}\sinh(\Delta|\tau|)\right)$$

Spectral function:

$$S_{v,v}(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\left\langle\langle v(t_1+\tau)v(t_1)\rangle_\xi\right\rangle_T$$

$$= \frac{kT}{m}\int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}e^{-\Gamma|\tau|}\left(\cosh(\Delta|\tau|) - \frac{\Gamma}{\Delta}\sinh(\Delta|\tau|)\right)$$

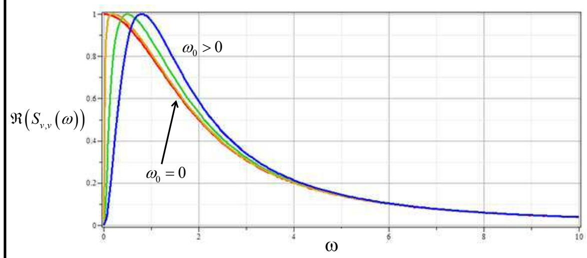
3/25/2014 PHY 770 Spring 2014 -- Lecture 16 15

Example of Brownian system -- continued

Spectral function:

$$S_{v,v}(\omega) = \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \left\langle \left\langle v(t_1+\tau)v(t_1) \right\rangle_\xi \right\rangle_T$$

$$= \frac{kT}{m} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} e^{-\Gamma|\tau|} \left(\cosh(\Delta|\tau|) - \frac{\Gamma}{\Delta} \sinh(\Delta|\tau|) \right)$$



Probability analysis of Brownian motion Fokker-Planck equation

Rather than analyzing single Brownian particles, it is convenient to analyze their probability density

$\rho(x, v, \xi, t)$: probability of finding the Brownian particle at time t with
position between x and $x + dx$
velocity between v and $v + dv$
random force ξ

Averaging over random forces:

$P(x, v, t) = \langle \rho(x, v, \xi, t) \rangle_\xi$

Fokker-Planck equation:

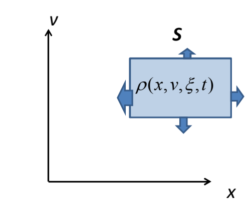
g comes from random force: $\langle \xi(t_1)\xi(t_2) \rangle_\xi = g\delta(t_1 - t_2)$

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left(\left(\frac{\gamma}{m} v - \frac{1}{m} F(x) \right) P \right) + \frac{g}{2m^2} \frac{\partial^2 P}{\partial v^2}$$

$F(x) = -\frac{\partial V}{\partial x}$ Force on particle due to potential $V(x)$

Probability analysis of Brownian motion Fokker-Plank equation

Justification of Fokker-Plank equation



Continuity condition

$$\frac{\partial}{\partial t} \iint dx dv \rho(x, v, \xi, t) = -\iint dx dv \left(\frac{\partial(\dot{x}\rho(x, v, \xi, t))}{\partial x} + \frac{\partial(\dot{v}\rho(x, v, \xi, t))}{\partial v} \right)$$

$$\frac{\partial(\rho(x, v, \xi, t))}{\partial t} = -\frac{\partial(\dot{x}\rho(x, v, \xi, t))}{\partial x} - \frac{\partial(\dot{v}\rho(x, v, \xi, t))}{\partial v}$$

Probability analysis of Brownian motion Fokker-Plank equation
Justification of Fokker-Plank equation

Langevin equation in presence of friction (γ) and potential force ($F(x) = -\nabla V(x)$):

$$\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) + \frac{1}{m}F(x) + \frac{1}{m}\xi(t) \qquad v(t) = \frac{dx(t)}{dt}$$

Continuity condition

$$\frac{\partial\left(\rho(x,v,\xi,t)\right)}{\partial t} = -\frac{\partial\left(v\rho(x,v,\xi,t)\right)}{\partial x} - \frac{\partial\left(\dot{v}\rho(x,v,\xi,t)\right)}{\partial v}$$

becomes:

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(v\rho)}{\partial x} + \frac{\partial}{\partial v}\left(\left(\frac{\gamma}{m}v(t) - \frac{1}{m}F(x) - \frac{1}{m}\xi(t)\right)\rho\right)$$

$$= -\frac{\partial(v\rho)}{\partial x} + \frac{\gamma}{m}\frac{\partial(v\rho)}{\partial v} - \frac{1}{m}F(x)\frac{\partial\rho}{\partial v} - \frac{1}{m}\xi(t)\frac{\partial\rho}{\partial v}$$

Probability analysis of Brownian motion Fokker-Plank equation
Justification of Fokker-Plank equation

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(v\rho)}{\partial x} + \frac{\gamma}{m}\frac{\partial(v\rho)}{\partial v} - \frac{1}{m}F(x)\frac{\partial\rho}{\partial v} - \frac{1}{m}\xi(t)\frac{\partial\rho}{\partial v}$$

$$\equiv -L_0\rho(t) - L_1\rho(t)$$

where $L_0 \equiv v\frac{\partial}{\partial x} - \frac{\gamma}{m} - \frac{\gamma}{m}v\frac{\partial}{\partial v} + \frac{1}{m}F(x)\frac{\partial}{\partial v}$

$L_1 \equiv \frac{1}{m}\xi(t)\frac{\partial}{\partial v}$

Define: $\rho(t) \equiv e^{-L_0 t}\sigma(t)$

$V(t) \equiv e^{L_0 t}L_1(t)e^{-L_0 t}$

$$\frac{\partial\sigma(t)}{\partial t} = -V(t)\sigma(t)$$

Formal solution for $\sigma(t)$: $\sigma(t) = \exp\left(-\int_0^t dt' V(t')\right)\sigma(0)$

Probability analysis of Brownian motion Fokker-Plank equation
Justification of Fokker-Plank equation

$\rho(t) \equiv e^{-L_0 t}\sigma(t)$

$$\sigma(t) = \exp\left(-\int_0^t dt' V(t')\right)\sigma(0)$$

Evaluation of equation using identity: $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\sigma(t) = \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\int_0^t dt' V(t')\right)^n\right)\sigma(0)$$

Recall: $V(t) \equiv e^{L_0 t}L_1(t)e^{-L_0 t}$ $\qquad L_1 \equiv \frac{1}{m}\xi(t)\frac{\partial}{\partial v}$

Averaging over random force:

$$\left\langle\sigma(t)\right\rangle_\xi = \left(\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left\langle\left(\int_0^t dt' V(t')\right)^{2n}\right\rangle_\xi\right)\sigma(0)$$

Probability analysis of Brownian motion Fokker-Plank equation
Justification of Fokker-Plank equation

Further analysis:

$$\langle \sigma(t) \rangle_\xi = \exp\left(\frac{1}{2} \int_0^t dt'' \int_0^t dt' \langle V(t'')V(t') \rangle_\xi \right) \sigma(0)$$

$$\frac{1}{2} \int_0^t dt'' \int_0^t dt' \langle V(t'')V(t') \rangle_\xi = \frac{g}{2m^2} \int_0^t dt'' \int_0^t dt' \delta(t''-t') e^{L_0 t''} \frac{\partial}{\partial v} e^{-L_0(t''-t')} \frac{\partial}{\partial v} e^{-L_0 t'}$$

$$= \frac{g}{2m^2} \int_0^t dt'' e^{L_0 t''} \frac{\partial^2}{\partial v^2} e^{-L_0 t''}$$

Differential equation for $\langle \sigma(t) \rangle_\xi$:

$$\frac{\partial \langle \sigma(t) \rangle_\xi}{\partial t} = \frac{g}{2m^2} e^{L_0 t} \frac{\partial^2}{\partial v^2} e^{-L_0 t} \langle \sigma(t) \rangle_\xi$$

Probability analysis of Brownian motion Fokker-Plank equation
Justification of Fokker-Plank equation

Differential equation for $\langle \sigma(t) \rangle_\xi$:

$$\frac{\partial \langle \sigma(t) \rangle_\xi}{\partial t} = \frac{g}{2m^2} e^{L_0 t} \frac{\partial^2}{\partial v^2} e^{-L_0 t} \langle \sigma(t) \rangle_\xi$$

Recall that: $\langle \rho(t) \rangle_\xi = e^{-L_0 t} \langle \sigma(t) \rangle_\xi$:

$$\frac{\partial \langle \rho(t) \rangle_\xi}{\partial t} = -L_0 \langle \rho(t) \rangle_\xi + \frac{g}{2m^2} \frac{\partial^2 \langle \rho(t) \rangle_\xi}{\partial v^2}$$

Recall: $P(x,v,t) = \langle \rho(x,v,\xi,t) \rangle_\xi$

Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -v \frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left(\left(\frac{\gamma}{m} v - \frac{1}{m} F(x) \right) P \right) + \frac{g}{2m^2} \frac{\partial^2 P}{\partial v^2}$$

Probability analysis of Brownian motion Fokker-Plank equation
Justification of Fokker-Plank equation

Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -v \frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left(\left(\frac{\gamma}{m} v - \frac{1}{m} F(x) \right) P \right) + \frac{g}{2m^2} \frac{\partial^2 P}{\partial v^2}$$

Define probability current:

$$\mathbf{J} = \hat{\mathbf{x}} v P - \hat{\mathbf{v}} \left(\frac{\gamma}{m} v P - \frac{1}{m} F(x) P + \frac{g}{2m^2} \frac{\partial P}{\partial v} \right)$$

$$\frac{\partial P}{\partial t} = -\nabla \cdot \mathbf{J} \qquad \text{where } \nabla \equiv \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{v}} \frac{\partial}{\partial v}$$