PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 7:

Start reading Chapter 3

Solution of Poisson equation in for special geometries –

A. Cylindrical

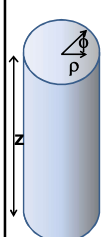
B. Spherical

Course schedule for Spring 2014

(Preliminary schedule -- subject to frequent adjustment.) Please note that makeup lectures (indicated in red) are scheduled for Tuesdays or Thursdays at 11 AM - 12:15 PM in Olin 107.

	Lecture date	JDJ Reading	Topic	Assign.	Due date
1	Wed: 01/15/2014	Chap. 1	Introduction, units and Poisson equation	#1	01/31/2014
2	Thu: 01/16/2014	Chap. 1	Electrostatic energy calculations	#2	01/31/2014
3	Fri: 01/17/2014	Chap. 1	Poisson equation and Green's theorm	#3	01/31/2014
	Mon: 01/20/2014		MLK Holiday - no class		
4	Wed: 01/22/2014	Chap. 1	Green's functions for cartesian coordinates	#4	01/31/2014
5	Thu: 01/23/2014	Chap. 1	Brief introduction to numerical methods	#5	01/31/2014
6	Fri: 01/24/2014	Chap. 2	Method of images	#6	01/31/2014
	Mon: 01/27/2014		NAWH out of town - no class		
	Wed: 01/29/2014		NAWH out of town - no class		
7	Fri: 01/31/2014	Chap. 3	Cylindrical and spherical geometries	#7	02/03/2014
8	Mon: 02/03/2014	Chap. 4	Multipole analysis of charge distributions	#8	02/03/2014

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence

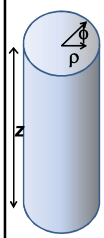


Laplace equation : $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

$$\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$$

Cylindrical geometry continued:



Laplace equation : $\nabla^2 \Phi = 0$

$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$

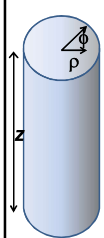
One possibility :

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right) R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry continued:



Laplace equation : $\nabla^2 \Phi = 0$

$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$

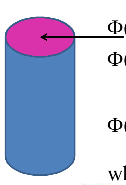
Another possibility :

$$\frac{d^2 Z}{dz^2} + k^2 Z = 0 \qquad \Rightarrow Z(z) = \sin(kz), \cos(kz), e^{\pm ikz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(-k^2 - \frac{m^2}{\rho^2}\right) R = 0 \qquad \Rightarrow I_m(k\rho), K_m(k\rho)$$

Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*



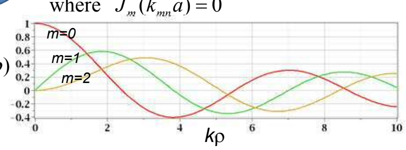
$\Phi(\rho,\phi,z=L) = V(\rho,\phi)$

$\Phi(\rho,\phi,z) = 0$ on all other boundaries

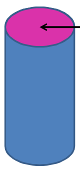
$$\Phi(\rho,\phi,z) = \sum_{n,m} A_{mn} J_m(k_{mn}\rho) \sinh(k_{mn} z) \sin(m\phi + \alpha_{mn})$$

where $J_m(k_{mn}a) = 0$

$J_m(k\rho)$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at *z=L*



$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

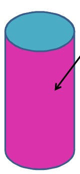
$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin(m\phi+\alpha_{mn})$$

If $V(\rho,\phi)$ is symmetric function of ϕ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\phi\cos(m\phi)\int_0^a \rho d\rho J_m(k_{mn}\rho)V(\rho,\phi)}{\sinh(k_{mn}L)\int_0^{2\pi} d\phi\cos^2(m\phi)\int_0^a \rho d\rho J_m{}^2(k_{mn}\rho)}$$

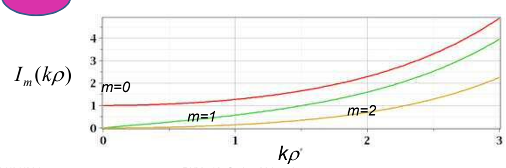
Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at ρ*=a*



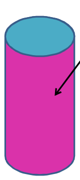
$$\Phi(\rho=a,\phi,z)=V(\phi,z)$$

$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin(m\phi+\alpha_{mn})$$



Solutions of Laplace equation inside cylindrical shape
Example with non-trivial boundary value at ρ*=a*



$$\Phi(\rho=a,\phi,z)=V(\phi,z)$$

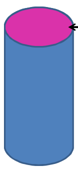
$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin(m\phi+\alpha_{mn})$$

If $V(z,\phi)$ is a symmetric function of ϕ so that $\alpha_{mn}=\pi/2$:

$$A_{mn}=\frac{\int_0^{2\pi} d\phi\cos(m\phi)\int_0^L dz\sin\left(\frac{n\pi z}{L}\right)V(z,\phi)}{I_m\left(\frac{n\pi a}{L}\right)\int_0^{2\pi} d\phi\cos^2(m\phi)\int_0^L dz\sin^2\left(\frac{n\pi z}{L}\right)}$$

Green's function for Dirchelet boundary value inside cylindar:



$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$

$\Phi(\rho=a,\phi,z)=0\,,\;\Phi(\rho,\phi,z=0)=0$

Expansion in terms of Bessel function zeros: $J_m(k_{mn}a)=0$

$$G(\rho,\rho',\phi,\phi',z,z')=\frac{8\pi}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}J_m(k_{mn}\rho)J_m(k_{mn}\rho')\sinh(k_{mn}z_<)\sinh(k_{mn}(L-z_>))}{k_{mn}(J_{m+1}(k_{mn}a))^2\sinh(k_{mn}L)}$$

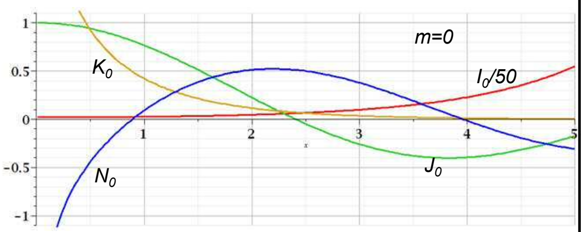
$$\Phi(\rho,\phi,z)=\frac{1}{4\pi\varepsilon_0}\int_V d\phi'\rho'd\rho'dz'\,G(\rho,\rho',\phi,\phi',z,z')\rho(\rho',\phi',z')+\frac{1}{4\pi}\int_{S;z'=L}d\phi'\rho'd\rho'\left.\frac{\partial G(\rho,\rho',\phi,\phi',z,z')}{\partial z'}\right|_{z'=L}V(\rho',\phi')$$

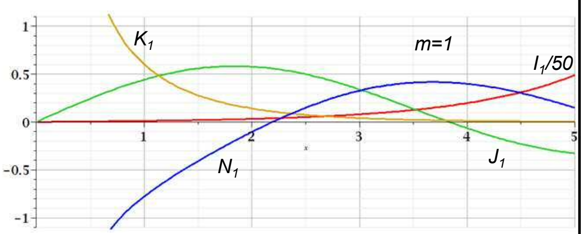
Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm iN_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$





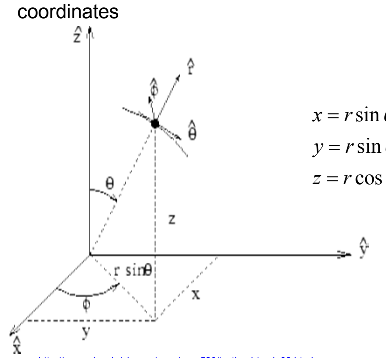
Some useful identities involving cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(1-\frac{m^2}{u^2}\right)\right)J_m(u)=0 \quad \text{for integer } m$$

Properties of Bessel functions in terms of zeros : x_{mn} $\quad J_m(x_{mn})=0$

$$\int_0^a \rho d\rho J_m(x_{mn}\rho/a)J_m(x_{mn'}\rho/a)=\frac{a^2}{2}\left(J_{m+1}(x_{mn})\right)^2\delta_{nn'}$$

Poisson and Laplace equation in spherical polar coordinates



$$x=r\sin\theta\cos\varphi$$
$$y=r\sin\theta\sin\varphi$$
$$z=r\cos\theta$$

http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html

Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r,\theta,\phi)$:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)\Phi=0$$

$$\Phi(r,\theta,\phi)=\sum_{lm}R_{lm}(r)Y_{lm}(\theta,\phi)$$

Spherical harmonic functions :

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)Y_{lm}(\theta,\phi)=-l(l+1)Y_{lm}(\theta,\phi)$$

Properties of spherical harmonic functions

$$Y_{lm}(\theta,\phi)=(-1)^m Y_{l(-m)}{}^*(\theta,\phi) \qquad \text{(standard Condon - Shortley convention)}$$

$$\int d\Omega Y_{lm}(\theta,\varphi)Y_{l'm'}{}^*(\theta,\varphi)\equiv\int \sin\theta\, d\theta\, d\varphi\, Y_{lm}(\theta,\varphi)Y_{l'm'}{}^*(\theta,\varphi)=\delta_{ll'}\delta_{mm'}$$

Completeness:

$$\sum_{lm} Y_{lm}(\theta,\varphi)Y_{lm}{}^*(\theta',\varphi')=\delta(\hat{\mathbf{r}}-\hat{\mathbf{r}}')\equiv\delta(\cos\theta-\cos\theta')\delta(\phi-\phi')$$

Relationship to Legendre polynomials:

$$Y_{l0}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}{}^*(\theta',\varphi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r\to\infty$:

$$\Phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$=\frac{1}{4\pi\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}{}^*(\theta',\varphi')\right)$$

Example -- continued

$$\Phi(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}{}^*(\theta',\varphi')\right)$$

Suppose: $\rho(\mathbf{r}')=\frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$

$$\int d\Omega' Y_{lm}{}^*(\theta',\varphi')=\sqrt{4\pi}\delta_{l0}\delta_{m0}$$

$$\Phi(\mathbf{r})=\frac{4\pi}{4\pi\varepsilon_0}\int_0^\infty r'^2\,dr'\frac{r_<^0}{r_>^1}\frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$$

$$=\frac{Q}{4\pi\varepsilon_0}\frac{\text{erf}(r/a)}{r}$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Elements of "proof":

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r_>\left(1+\left(\frac{r_<}{r_>}\right)^2-2\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right)^{1/2}}=$$

$$=\frac{1}{r_>}\left(1+\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'+\left(\frac{r_<}{r_>}\right)^2\left(\frac{3}{2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2-\frac{1}{2}\right)+\cdots\right)$$

$$=\frac{1}{r_>}\left(\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\right)$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^*(\theta',\varphi')$$

Elements of "proof" -- continued:

Sum rule for spherical harmonics:

$$P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}(\hat{\mathbf{r}})\,Y_{lm}^*(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}}=\hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')=1$

$$\Rightarrow\frac{4\pi}{2l+1}\sum_{m=-l}^{l}|Y_{lm}(\hat{\mathbf{r}})|^2=1$$

Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}})=\frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm1)}(\hat{\mathbf{r}})=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}})=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2(\pm2)}(\hat{\mathbf{r}})=\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

$$Y_{2(\pm1)}(\hat{\mathbf{r}})=\mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}})=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$$