

**PHY 712 Electrodynamics**  
**10-10:50 AM MWF Olin 107**

**Plan for Lecture 30:**  
**Special Topics in Electrodynamics:**  
**Cherenkov radiation**  
**(discussed in Jackson Chapter 13)**

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Fri: 03/07/2014	AP's Meeting	Take-home exam (no class meeting)		
Mon: 03/10/2014	Spring Break			
Wed: 03/12/2014	Spring Break			
Fri: 03/14/2014	Spring Break			
19 Mon: 03/17/2014	Chap. 8	Wave guides Take-home exam due	#18	3/21/2014
20 Wed: 03/19/2014	Chap. 9	Sources of Electromagnetic Waves	#19	3/21/2014
21 Fri: 03/21/2014	Chap. 9	Sources of Electromagnetic Waves	#20	3/28/2014
22 Mon: 03/24/2014	Chap. 11	Special Theory of Relativity	#21	3/28/2014
23 Wed: 03/26/2014	Chap. 11	Special Theory of Relativity	#22	3/28/2014
24 Fri: 03/28/2014	Chap. 11	Special Theory of Relativity	#23	4/04/2014
26 Mon: 03/31/2014	Chap. 14	Radiation from moving charges	#24	4/04/2014
26 Wed: 04/02/2014	Chap. 14	Radiation from moving charges	#25	4/04/2014
27 Fri: 04/04/2014	Chap. 14	Radiation from moving charges	#26	4/11/2014
28 Mon: 04/07/2014	Chap. 14	Radiation from moving charges	#27	4/11/2014
29 Wed: 04/09/2014	Chap. 15	Radiation due to collision processes	#28	4/14/2014
30 Fri: 04/11/2014	Chap. 16	Cherenkov radiation	#29	4/14/2014
31 Mon: 04/14/2014				
32 Wed: 04/16/2014				
Fri: 04/18/2014	Good Friday			
33 Mon: 04/21/2014				
34 Wed: 04/23/2014				
35 Fri: 04/25/2014				
Mon: 04/28/2014		Presentations Part I		
Wed: 04/30/2014		Presentations Part II		

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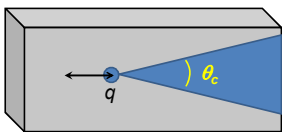
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References for notes: Glenn S. Smith, *An Introduction to Electromagnetic Radiation* (Cambridge UP, 1997), Andrew Zangwill, *Modern Electrodynamics* (Cambridge UP, 2013)

Cherenkov radiation  
 Discovered ~1930; bluish light emitted by energetic charged particles traveling within dielectric materials



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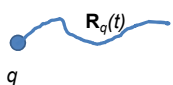
Maxwell's potential equations within a material having permittivity and permeability (Lorentz gauge; cgs Gaussian units)

$$\nabla^2 \Phi - \mu\epsilon \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{4\pi}{\epsilon} \rho$$

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi\mu}{c} \mathbf{J}$$

Source: charged particle moving on trajectory  $\mathbf{R}_q(t)$ :

$$\rho(\mathbf{r}, t) = q \delta(\mathbf{r} - \mathbf{R}_q(t))$$

$$\mathbf{J}(\mathbf{r}, t) = q \dot{\mathbf{R}}_q(t) \delta(\mathbf{r} - \mathbf{R}_q(t))$$


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Liénard-Wiechert potential solutions:

$$\Phi(\mathbf{r}, t) = \frac{q}{\epsilon} \frac{1}{R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r)}$$

$$\mathbf{A}(\mathbf{r}, t) = q\mu \frac{\boldsymbol{\beta}_n}{R(t_r) - \boldsymbol{\beta}_n \cdot \mathbf{R}(t_r)}$$

$$\mathbf{R}(t_r) \equiv \mathbf{r} - \mathbf{R}_q(t_r)$$

$$\boldsymbol{\beta}_n(t_r) \equiv \frac{\dot{\mathbf{R}}_q(t_r)}{c_n} \quad c_n \equiv \sqrt{\mu\epsilon} \quad c \equiv \frac{c}{n}$$

$$t_r = t - \frac{R(t_r)}{c_n}$$

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Some algebra

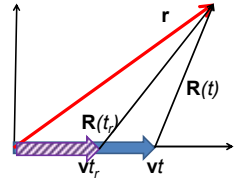
$$\mathbf{R}(t) = \mathbf{r} - \mathbf{v}t$$

$$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t - t_r)$$

$$(t - t_r)c_n = R(t_r) = |\mathbf{R}(t) + \mathbf{v}(t - t_r)|$$

Quadratic equation for  $(t - t_r)c_n$ :

$$((t - t_r)c_n)^2 = R^2(t) + 2\mathbf{R}(t) \cdot \boldsymbol{\beta}_n (t - t_r)c_n + \beta_n^2 ((t - t_r)c_n)^2$$

$$(t - t_r)c_n = \frac{-\mathbf{R}(t) \cdot \boldsymbol{\beta}_n \pm \sqrt{(\mathbf{R}(t) \cdot \boldsymbol{\beta}_n)^2 - (\beta_n^2 - 1)R^2(t)}}{\beta_n^2 - 1}$$


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$\mathbf{R}(t_r) = \mathbf{r} - \mathbf{v}t_r = \mathbf{R}(t) + \mathbf{v}(t - t_r)$   
 $(t - t_r)c_n = R(t)$   
 $R(t_r) - \mathbf{R}(t_r) \cdot \hat{\boldsymbol{\beta}}_n =$   
 $(t - t_r)c_n(1 - \beta_n^2) - \mathbf{R}(t) \cdot \hat{\boldsymbol{\beta}}_n$

$$R(t_r) = \frac{-\mathbf{R}(t) \cdot \hat{\boldsymbol{\beta}}_n \pm \sqrt{(\mathbf{R}(t) \cdot \hat{\boldsymbol{\beta}}_n)^2 - (\beta_n^2 - 1)R^2(t)}}{\beta_n^2 - 1}$$

$$R(t_r) = \frac{R(t)}{\beta_n^2 - 1} \left( -\beta_n \cos \theta \pm \sqrt{1 - \beta_n^2 \sin^2 \theta} \right)$$

$$R(t_r) - \mathbf{R}(t_r) \cdot \hat{\boldsymbol{\beta}}_n = \mp R(t) \sqrt{1 - \beta_n^2 \sin^2 \theta}$$

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Liénard-Wiechert potentials:

$$\Phi(\mathbf{r}, t) = \pm \frac{q}{\epsilon} \frac{1}{R(t) \sqrt{1 - \beta_n^2 \sin^2 \theta}}$$

$$\mathbf{A}(\mathbf{r}, t) = \pm q\mu \frac{\hat{\boldsymbol{\beta}}_n}{R(t) \sqrt{1 - \beta_n^2 \sin^2 \theta}}$$

For  $\beta_n > 1$ , range of  $\theta$  is limited:

$$R(t_r) = \frac{R(t)}{\beta_n^2 - 1} \left( -\beta_n \cos \theta \pm \sqrt{1 - \beta_n^2 \sin^2 \theta} \right) \geq 0$$

$$\Rightarrow \frac{\pi}{2} \leq \theta \leq \sin^{-1} \left( \frac{1}{\beta_n} \right)$$

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Physical fields for  $\beta_n > 1$

$$\frac{\pi}{2} \leq \theta \leq \sin^{-1} \left( \frac{1}{\beta_n} \right)$$

Define  $\cos \theta_c \equiv \sqrt{1 - \frac{1}{\beta_n^2}}$

$$\Rightarrow \cos \theta \leq \cos \theta_c$$

$$\Phi(\mathbf{r}, t) = \frac{2q}{\epsilon} \frac{1}{R(t) \sqrt{1 - \beta_n^2 \sin^2 \theta}} \Theta(\cos \theta_c - \cos \theta(t))$$

$$\mathbf{A}(\mathbf{r}, t) = 2q\mu \frac{\hat{\boldsymbol{\beta}}_n}{R(t) \sqrt{1 - \beta_n^2 \sin^2 \theta}} \Theta(\cos \theta_c - \cos \theta(t))$$

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Physical fields for  $\beta > 1$

$$\Phi(\mathbf{r}, t) = \frac{2q}{\epsilon} \frac{1}{R(t)\sqrt{1-\beta_n^2 \sin^2 \theta}} \Theta(\cos \theta_c - \cos \theta(t))$$

$$\mathbf{A}(\mathbf{r}, t) = 2q\mu \frac{\hat{\mathbf{R}}_n}{R(t)\sqrt{1-\beta_n^2 \sin^2 \theta}} \Theta(\cos \theta_c - \cos \theta(t))$$

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi - \frac{1}{c_n} \frac{\partial \mathbf{A}}{\partial t} \quad \mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{2q}{\epsilon} \frac{\hat{\mathbf{R}}}{(R(t))^2 \sqrt{1-\beta_n^2 \sin^2 \theta}} \times \left( \frac{\beta_n^2 - 1}{1-\beta_n^2 \sin^2 \theta} \Theta(\cos \theta_c - \cos \theta(t)) + \sqrt{\beta_n^2 - 1} \delta(\cos \theta_c - \cos \theta(t)) \right)$$

$$\mathbf{B}(\mathbf{r}, t) = -\beta_n \sin \theta (\hat{\theta} \times \mathbf{E}(\mathbf{r}, t))$$

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Intermediate steps:

$$\frac{d\theta}{dt} = \frac{v \sin \theta}{R} \quad \frac{dR}{dt} = -v \cos \theta$$

Using instantaneous polar coordinates:  $\nabla \equiv \hat{\mathbf{R}} \frac{\partial}{\partial R} + \hat{\theta} \frac{1}{R} \frac{\partial}{\partial \theta}$

$$\nabla \Theta(\cos \theta_c - \cos \theta(t)) = \delta(\cos \theta_c - \cos \theta(t)) \frac{\sin \theta(t)}{R(t)} \hat{\theta}$$

$$\frac{\partial \Theta(\cos \theta_c - \cos \theta(t))}{\partial t} = \delta(\cos \theta_c - \cos \theta(t)) \frac{v \sin^2 \theta(t)}{R(t)}$$

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Some details: Cherenkov radiation observed near the angle  $\theta_c$  at time  $t=t_c+\Delta t$

$$\cos \theta_c - \cos \theta(t) \approx \frac{c_n \Delta t}{\beta_n R_c}$$

$$1 - \beta_n \sin^2 \theta(t) \approx \frac{2c_n \Delta t \sqrt{\beta_n^2 - 1}}{R_c}$$

When the dust clears ....

$$\frac{d^2 I}{d\omega d\ell} \propto \left( 1 - \frac{c_n^2}{v^2} \right) \omega$$

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Some details: Cherenkov radiation observed near the angle  $\theta_c$

$\mathbf{R}(t) = \mathbf{r} - \mathbf{v}t$   
 $\sin \theta_c = \frac{c_n}{v}$   
 $\pi \geq \theta(t) \geq \theta_c$

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Cherenkov radiation observed near the angle  $\theta_c$  -- continued

$\cos \theta_c - \cos \theta(\Delta t) \approx \sin \theta_c \theta(\Delta t)$   
 $\approx \frac{c_n \Delta t}{R_c}$   
 $1 - \beta_n^2 \sin^2 \theta(t) \approx 2\sqrt{\beta_n^2 - 1} \frac{c_n \Delta t}{R_c}$

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Cherenkov radiation observed near the angle  $\theta_c$  -- continued

$$\mathbf{E}(\mathbf{r}, t) = \frac{2q}{\epsilon} \frac{\hat{\mathbf{R}}}{(R(t))^2 \sqrt{1 - \beta_n^2 \sin^2 \theta}} \times$$

$$\left( \frac{\beta_n^2 - 1}{1 - \beta_n^2 \sin^2 \theta} \Theta(\cos \theta_c - \cos \theta(t)) + \sqrt{\beta_n^2 - 1} \delta(\cos \theta_c - \cos \theta(t)) \right)$$

$$\mathbf{B}(\mathbf{r}, t) = -\beta_n \sin \theta (\hat{\theta} \times \mathbf{E}(\mathbf{r}, t))$$

Estimates at  $t = t_c + \Delta t$

$$\mathbf{E}(\mathbf{r}, t) \approx -\frac{2q}{\epsilon} \hat{\mathbf{R}}_c \frac{(\beta_n^2 - 1)^{1/4}}{(2c_n^3 R_c)^{1/2}} \left[ (\Delta t)^{-1/2} \delta(\Delta t) - \frac{1}{2} (\Delta t)^{-3/2} \Theta(\Delta t) \right]$$

$$\mathbf{B}(\mathbf{r}, t) = -(\hat{\theta}(0) \times \mathbf{E}(\mathbf{r}, t))$$

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Cherenkov radiation observed near the angle  $\theta_c$  -- continued

Spectral analysis:

$$\begin{aligned} \tilde{\mathbf{E}}(\omega) &= -\frac{2q}{\epsilon} \hat{\mathbf{R}}_c \frac{(\beta_n^2 - 1)^{1/4}}{(2c_n^3 R_c)^{1/2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \left[ t^{-1/2} \delta(t) - \frac{1}{2} t^{-3/2} \Theta(t) \right] e^{i\omega t} \\ &= -i\omega \frac{2q}{\epsilon} \hat{\mathbf{R}}_c \frac{(\beta_n^2 - 1)^{1/4}}{(2c_n^3 R_c)^{1/2}} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dt t^{-1/2} e^{i\omega t} \\ &= \frac{q}{\epsilon} \hat{\mathbf{R}}_c \frac{(\beta_n^2 - 1)^{1/4}}{(2c_n^3 R_c)^{1/2}} (1-i) \sqrt{\omega} \end{aligned}$$

Spectral intensity:  $\frac{d^2 I}{d\Omega d\omega} \propto |\tilde{\mathbf{E}}(\omega)|^2 = \frac{q^2 (\beta_n^2 - 1)^{1/2}}{\epsilon^2 c_n^3 R_c} \omega$

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Cherenkov radiation emitted by the core of the Reed Research Reactor located at Reed College in Portland, Oregon, U.S. *Cherenkov radiation*. Photograph. *Encyclopædia Britannica Online*. Web. 12 Apr. 2013. <http://www.britannica.com/EBchecked/media/174732>

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