PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 22:

- **Comments on Mid-Term Exam**
- **Start reading Chap. 11**

  A. Equations in cgs (Gaussian) units

  B. Special theory of relativity

  C. Lorentz transformation relations

8	Mon: 02/03/2014	Chap. 4	Multipole analysis of charge distributions	#8	02/05/2014
9	Wed: 02/05/2014	Chap. 4	Dipoles and dielectrics	#9	02/07/2014
10	Fri: 02/07/2014	Chap. 4	Dipoles and dielectrics	#10	02/10/2014
11	Mon: 02/10/2014	Chap. 5	Magnetostatics	#11	02/12/2014
12	Wed: 02/12/2014	Chap. 5	Magnetostatics	#12	02/14/2014
	Fri: 02/14/2014		Class cancelled because of weather		
13	Mon: 02/17/2014	Chap. 5	Magnetostatics	#13	02/28/2014
14	Mon: 02/17/2014	Chap. 6	Maxwell's equations	#14	02/28/2014
15	Wed: 02/19/2014	Chap. 6	Electromagnetic energy and force	#15	02/28/2014
16	Fri: 02/21/2014	Chap. 7	Electromagnetic plane waves	#16	02/28/2014
17	Fri: 02/21/2014	Chap. 7	Dynamic dielectric media and their effects	#17	02/28/2014
	Mon: 02/24/2014		No class -- NAWH out of town		
	Wed: 02/26/2014		No class -- NAWH out of town		
18	Fri: 02/28/2014	Chap. 7	Complex dielectrics; TEM modes		
	Mon: 03/03/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Wed: 03/05/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Fri: 03/07/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Mon: 03/10/2014	*Spring Break*			
	Wed: 03/12/2014	*Spring Break*			
	Fri: 03/14/2014	*Spring Break*			
19	Mon: 03/17/2014	Chap. 8	Wave guides;Take-home exam due	#18	3/21/2014
20	Wed: 03/19/2014	Chap. 9	Sources of Electromagnetic Waves	#19	3/21/2014
21	Fri: 03/21/2014	Chap. 9	Sources of Electromagnetic Waves	#20	3/28/2014
22	Mon: 03/24/2014	Chap. 11	Special Theory of Relativity	#21	3/28/2014

Comments on Mid-Term Exam

1. Consider the following one-dimensional charge distribution in vacuum:

$$\rho(x) = \begin{cases} 0 & x < -a \\ \rho_0 \sin(\pi x/a) & -a \le x \le a \\ 0 & x > a \end{cases}$$

where ρ_0 is a constant having the units of charge per unit volume and a is a length constant. It is assumed that the charge density and all related fields are uniform in the y and z directions.

(a) Find the electrostatic potential as a function of x for all $-\infty \le x \le \infty$ due to this charge distribution, assuming the boundary conditions

$$\left.\frac{d\Phi(x)}{dx}\right|_{-\infty} = 0 \quad \text{and} \quad \left.\frac{d\Phi(x)}{dx}\right|_{\infty} = 0.$$

(b) Determine the value of $\Phi(x = \infty) - \Phi(x = -\infty)$.

(c) Determine the corresponding x-component of the electrostatic field as a function of x:

$$E_x(x) = -\frac{d\Phi(x)}{dx}.$$

(d) Find the corresponding electrostatic energy W and electrostatic energy density $w(x)$ discussed in Eqs. 1.54 and 1.55 of your textbook.

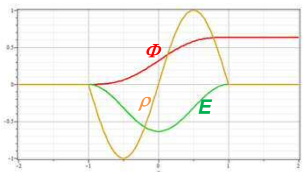
Use Maple or another program to generate plots of your results.

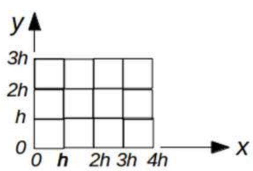
Green's function solution:

$$\Phi(x) = \begin{cases} 0 & \text{for } x < -a \\ \frac{\rho_0}{\varepsilon_0}\int_{-a}^{a} x_< \sin\left(\frac{\pi x'}{a}\right) dx = \frac{\rho_0 a^2}{\varepsilon_0 \pi}\left(\frac{1}{\pi}\sin\left(\frac{\pi x}{a}\right) + 1 + \frac{x}{a}\right) & \text{for } -a < x < a \\ \frac{2\rho_0 a^2}{\varepsilon_0 \pi} & \text{for } x > a \end{cases}$$

$$E(x) = \begin{cases} 0 & \text{for } x < -a \\ -\frac{\rho_0 a^2}{\varepsilon_0 \pi}\left(\cos\left(\frac{\pi x}{a}\right) + 1\right) & \text{for } -a < x < a \\ 0 & \text{for } x > a \end{cases}$$

$$W = \int_{-a}^{a} w(x)dx = \frac{\varepsilon_0}{2}\int_{-a}^{a} |E(x)|^2 dx = \frac{3\rho_0^2 a^3}{2\pi^2 \varepsilon_0}$$





2.

Consider the area shown in the diagram above where $0 \le x \le 4h$ and $0 \le y \le 3h$. A charge density within this area is given by

$$\rho(x,y) = \rho_0 \sin\left(\frac{\pi x}{4h}\right)\sin\left(\frac{\pi y}{3h}\right).$$

In this problem, you will find the corresponding electrostatic potential $\Phi(x,y,)$ with the boundary conditions $\Phi(0,y) = \Phi(x,0) = \Phi(4h,y) = \Phi(x,3h) = 0$, using 3 different methods.

(a) First find analytic form for $\Phi(x,y)$ using Green's functions or other methods of your choice.

Recall from Lecture 4: Green's function for 3-d Poisson equation in Cartesian coordinates:

$$\nabla^2\Phi(\mathbf{r}) \equiv \frac{\partial^2\Phi(\mathbf{r})}{\partial x^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial y^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial z^2} = -\rho(\mathbf{r})/\epsilon_0. \quad (39)$$

The orthogonal function expansion method can easily be extended to two and three dimensions. For example if $\{u_n(x)\}$, $\{v_n(x)\}$, and $\{w_n(x)\}$ denote the complete functions in the x, y, and z directions respectively, then the three dimensional Green's function can be written:

$$G(x,x',y,y',z,z') = 4\pi\sum_{lmn}\frac{u_l(x)u_l(x')v_m(y)v_m(y')w_n(z)w_n(z')}{\alpha_l + \beta_m + \gamma_n}, \quad (40)$$

where

$$\frac{d^2}{dx^2}u_l(x) = -\alpha_l u_l(x), \quad \frac{d^2}{dy^2}v_m(x) = -\beta_m v_m(y), \quad \text{and} \quad \frac{d^2}{dz^2}w_n(z) = -\gamma_n w_n(z).$$

In 2-dimensions:

$$\nabla^2\Phi(\mathbf{r}) \equiv \frac{\partial^2\Phi(\mathbf{r})}{\partial x^2} + \frac{\partial^2\Phi(\mathbf{r})}{\partial y^2} = -\rho(\mathbf{r})/\epsilon_0$$

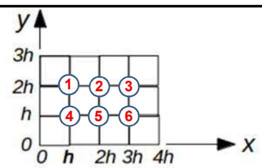
$$G(x,x',y,y') = 4\pi\sum_{lm}\frac{u_l(x)u_l(x')v_m(y)v_m(y')}{\alpha_l+\beta_m}$$

In our case: $u_l(0)=u_l(4h) \Rightarrow u_l(x) = \sqrt{\frac{1}{2h}}\sin\left(\frac{l\pi x}{4h}\right) \quad \alpha_l = \left(\frac{l\pi}{4h}\right)^2$

$$v_m(0)=v_m(3h) \Rightarrow v_m(x) = \sqrt{\frac{2}{3h}}\sin\left(\frac{m\pi x}{3h}\right) \quad \beta_m = \left(\frac{m\pi}{3h}\right)^2$$

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^{4h}dx'\int_0^{3h}dy' G(x,x',y,y')\rho(x',y')$$

$$= \frac{\rho_0}{\varepsilon_0}\frac{144h^2}{25\pi^2}\sin\left(\frac{\pi x}{4h}\right)\sin\left(\frac{\pi y}{3h}\right)$$



$$\Phi(\mathbf{r}) = \frac{\rho_0}{\varepsilon_0}\frac{144h^2}{25\pi^2}\sin\left(\frac{\pi x}{4h}\right)\sin\left(\frac{\pi y}{3h}\right)$$

$$\Phi(x_1,y_1) = \Phi_1 = \frac{\rho_0}{\varepsilon_0}\frac{h^2}{\pi^2}3.527265230$$

$$\Phi(x_2,y_2) = \Phi_2 = \frac{\rho_0}{\varepsilon_0}\frac{h^2}{\pi^2}4.988306327$$

2.

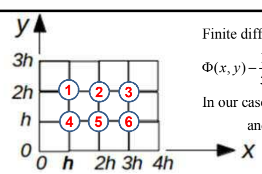
Consider the area shown in the diagram above where $0 \le x \le 4h$ and $0 \le y \le 3h$. A charge density within this area is given by

$$\rho(x,y) = \rho_0 \sin\left(\frac{\pi x}{4h}\right)\sin\left(\frac{\pi y}{3h}\right).$$

In this problem, you will find the corresponding electrostatic potential $\Phi(x,y,)$ with the boundary conditions $\Phi(0,y)=\Phi(x,0)=\Phi(4h,y)=\Phi(x,3h)=0$, using 3 different methods.

(b) Now find the numerical approximation to $\Phi(x,y)$ using the finite difference method discussed in class and in your text book. Compare your numerical results with the analytic results.

(c) Finally find the numerical approximation to $\Phi(x,y)$ using the finite element method discussed in class and in your text book. Compare your numerical results with the analytic results.



Finite difference equation for 2-dim cartesian case:

$$\Phi(x,y) - \frac{1}{5}S_A - \frac{1}{20}S_B = \frac{3h^2}{10\varepsilon_0}\rho(x,y) + \frac{h^4}{40\varepsilon_0}\nabla^2\rho(x,y)$$

In our case: $\Phi(x_1,y_1) \equiv \Phi_1 = \Phi_3 = \Phi_4 = \Phi_6$

and $\Phi_2 = \Phi_5$

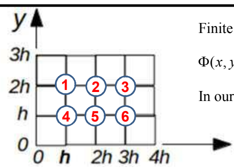
Unique equations:

$$\Phi_1 - \frac{1}{5}(\Phi_2+\Phi_4) - \frac{1}{20}(\Phi_5) = \Phi_1 - \frac{1}{5}(\Phi_2+\Phi_1) - \frac{1}{20}(\Phi_2) = \frac{3h^2}{10\varepsilon_0}\rho_1 + \frac{h^4}{40\varepsilon_0}\nabla^2\rho_1$$

$$\Phi_2 - \frac{1}{5}(\Phi_1+\Phi_3+\Phi_5) - \frac{1}{20}(\Phi_4+\Phi_6) = \Phi_2 - \frac{1}{5}(2\Phi_1+\Phi_2) - \frac{1}{20}(2\Phi_1) = \frac{3h^2}{10\varepsilon_0}\rho_2 + \frac{h^4}{40\varepsilon_0}\nabla^2\rho_2$$

$$\frac{3h^2}{10\varepsilon_0}\rho(x,y) + \frac{h^4}{40\varepsilon_0}\nabla^2\rho(x,y) = \frac{h^2}{\pi^2\varepsilon_0}\left(\frac{3}{10} - \frac{1}{40}\frac{25\pi^2}{144}\right)\pi^2\rho(x,y)$$

$$\frac{3h^2}{10\varepsilon_0}\rho_1 + \frac{h^4}{40\varepsilon_0}\nabla^2\rho_1 = \frac{h^2\rho_0}{\pi^2\varepsilon_0}1.554261748 \qquad \frac{3h^2}{10\varepsilon_0}\rho_2 + \frac{h^4}{40\varepsilon_0}\nabla^2\rho_2 = \frac{h^2\rho_0}{\pi^2\varepsilon_0}2.198058045$$



Finite difference equation for 2-dim cartesian case:

$$\Phi(x,y) - \frac{1}{5}S_A - \frac{1}{20}S_B = \frac{3h^2}{10\varepsilon_0}\rho(x,y) + \frac{h^4}{40\varepsilon_0}\nabla^2\rho(x,y)$$

In our case: $\Phi(x_1,y_1) \equiv \Phi_1 = \Phi_3 = \Phi_4 = \Phi_6$

and $\Phi_2 = \Phi_5$

Linear equations:

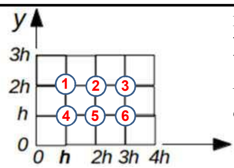
$$\begin{pmatrix} 4/5 & -1/4 \\ -1/2 & 4/5 \end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{\rho_0}{\varepsilon_0}\frac{h^2}{\pi^2}\begin{pmatrix} 1.554261748 \\ 2.198058045 \end{pmatrix}$$

Finite difference:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{\rho_0}{\varepsilon_0}\frac{h^2}{\pi^2}\begin{pmatrix} 3.481406 \\ 4.923451 \end{pmatrix}$$

Analytic:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{\rho_0}{\varepsilon_0}\frac{h^2}{\pi^2}\begin{pmatrix} 3.527265 \\ 4.988306 \end{pmatrix}$$



Finite element equation for 2-dim cartesian case:

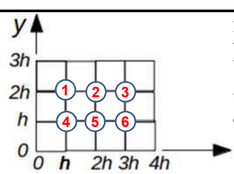
$$\sum_{ij} M_{kl,ij}\psi_{ij} = G_{kl}, \quad \text{where}$$

$$M_{kl,ij} \equiv \int dx \int dy \nabla\phi_{kl}(x,y)\cdot\nabla\phi_{ij}(x,y) \quad \text{and}$$

$$G_{kl} \equiv \int dx \int dy \phi_{kl}(x,y)\,\rho(x,y)/\varepsilon_0$$

$$M_{kl,ij} = \begin{cases} \frac{8}{3} & \text{for } k=i \text{ and } l=j \\ -\frac{1}{3} & \text{for } k-i=\pm1 \text{ and/or } l-j=\pm1 \\ 0 & \text{otherwise} \end{cases}$$

$$G_{kl} = \int dx \int dy \phi_{kl}(x,y)\,\rho(x,y)/\varepsilon_0 \approx \left(\rho(x_k,y_l)/\varepsilon_0\right)\int dx \int dy \phi_{kl}(x,y)$$
$$= h^2\rho(x_k,y_l)/\varepsilon_0$$



Finite element equation for 2-dim cartesian case:

$$\sum_{ij} M_{kl,ij}\psi_{ij} = G_{kl}, \quad \text{where}$$

$$M_{kl,ij} \equiv \int dx \int dy \nabla\phi_{kl}(x,y)\cdot\nabla\phi_{ij}(x,y) \quad \text{and}$$

$$G_{kl} \equiv \int dx \int dy \phi_{kl}(x,y)\,\rho(x,y)/\varepsilon_0$$

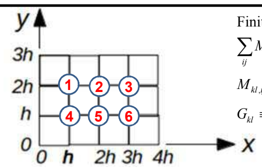
For our case:

$$\frac{8}{3}\Phi_1 - \frac{1}{3}\left(\Phi_2 + \Phi_4 + \Phi_5\right) = \frac{8}{3}\Phi_1 - \frac{1}{3}\left(2\Phi_2 + \Phi_1\right) = \frac{h^2}{\varepsilon_0}\rho_1$$

$$\frac{8}{3}\Phi_2 - \frac{1}{3}\left(\Phi_1 + \Phi_4 + \Phi_5 + \Phi_6 + \Phi_3\right) = \frac{8}{3}\Phi_2 - \frac{1}{3}\left(4\Phi_1 + \Phi_2\right) = \frac{h^2}{\varepsilon_0}\rho_2$$

$$\begin{pmatrix} 7/3 & -2/3 \\ -4/3 & 7/3 \end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{\rho_0 h^2}{\varepsilon_0\pi^2}\begin{pmatrix} 6.043873688 \\ 8.547328140 \end{pmatrix}$$

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{\rho_0 h^2}{\varepsilon_0\pi^2}\begin{pmatrix} 4.346471 \\ 6.146838 \end{pmatrix} \quad \text{(rather poor result)}$$



Finite element equation for 2-dim cartesian case:

$$\sum_{ij} M_{kl,ij}\psi_{ij} = G_{kl}, \qquad \text{where}$$

$$M_{kl,ij} \equiv \int dx \int dy \nabla\phi_{kl}(x,y)\cdot\nabla\phi_{ij}(x,y) \quad \text{and}$$

$$G_{kl} \equiv \int dx \int dy \phi_{kl}(x,y)\,\rho(x,y)/\varepsilon_0$$

Better treatment of G_{kl}:

$$G_1 = \frac{\rho_0 h^2}{\varepsilon_0 \pi^2} 5.233806864 \qquad G_2 = \frac{\rho_0 h^2}{\varepsilon_0 \pi^2} 7.401720668$$

$$\begin{pmatrix} 7/3 & -2/3 \\ -4/3 & 7/3 \end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{\rho_0 h^2}{\varepsilon_0 \pi^2}\begin{pmatrix} 5.233806864 \\ 7.401720668 \end{pmatrix}$$

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{\rho_0 h^2}{\varepsilon_0 \pi^2}\begin{pmatrix} 3.763909 \\ 5.322971 \end{pmatrix} \quad \text{(pretty good result)}$$

3. Consider a 3-dimensional charge density expressed in spherical polar coordinates

$$\rho(r,\theta,\phi) = Q_0 e^{-r/a} + Q_1 \frac{r}{b} e^{-r/b}\cos\theta,$$

where Q_0 and Q_1 are constants having the units of charge per unit volume and a and b are length parameters.

(a) Find an analytic expression for the electrostatic potential as a function of position.

(b) Analyze the potential for $r \to 0$ and describe its qualitative features.

(c) Analyze the potential for $r \to \infty$ and describe its qualitative features. In particular, can you write the potential in the form

$$\Phi(\mathbf{r}) \overset{r\to\infty}{=} \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \dots\right),$$

and if so, explain.

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\,\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$

For our case

$$\rho_{00}(r) = \sqrt{4\pi}Q_0 e^{-r/a} \qquad \rho_{10}(r) = \sqrt{\frac{4\pi}{3}}Q_1\frac{r}{b}e^{-r/b}$$

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\sum_{lm}\frac{1}{2l+1}Y_{lm}(\theta,\varphi)\left(\frac{1}{r^{l+1}}\int_0^r r'^{2+l}\,dr'\,\rho_{lm}(r') + r^l\int_r^\infty r'^{1-l}\,dr'\rho_{lm}(r')\right)$$

Evaluating the expressions with the help of Maple:

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\left[\frac{2a^3Q_0}{r}\left(1-e^{-r/a}\left(1+\frac{r}{2a}\right)\right) + \frac{24b^4Q_1}{r^2}\cos\theta\left(1-e^{-r/b}\left(1+\frac{r}{b}+\frac{r^2}{2b^2}+\frac{r^2}{8b^3}\right)\right)\right]$$

Also with the help of Maple, can evaluate the limit $r \to 0$:

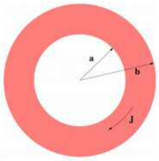
$$\Phi(\mathbf{r}) \approx \frac{1}{\varepsilon_0}\left[Q_0\left(a^2 + O(r^2)\right) + Q_1\cos\theta\left(br + O(r^3)\right)\right]$$

For the limit $r \to \infty$:

$$\Phi(\mathbf{r}) \approx \frac{1}{\varepsilon_0}\left[\frac{2a^3Q_0}{r} + \frac{24b^4Q_1}{r^2}\cos\theta\right]$$

Identify monopole charge: $q = 8\pi a^3 Q_0$

Identify dipole: $p = 96\pi b^4 Q_1$



4.

The figure above shows the cross section of a magnetostatic solenoid which is uniform in the $\hat{\mathbf{z}}$ direction (perpendicular to the page). The current flows in the azimuthal $\hat{\phi}$ direction; specifically the current density is given in cylindrical coordinates by:

$$\mathbf{J} = \begin{cases} J_0\hat{\phi} & a \le \rho \le b \\ 0 & \text{otherwise.} \end{cases} \qquad (1)$$

Here J_0 is a constant, a and b denote the inner and outer diameters of the cylinder, respectively, and $\hat{\phi} = -\sin(\phi)\hat{\mathbf{x}} + \cos(\phi)\hat{\mathbf{y}}$.

(a) Show that the vector potential $\mathbf{A}$ for this system can be written as

$$\mathbf{A} = f(\rho)\hat{\phi}, \qquad (2)$$

where the scalar function $f(\rho)$ satisfies the equation

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2}\right] f(\rho) = \begin{cases} -\mu_0 J_0 & a \le \rho \le b \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

(b) Find the function $f(\rho)$ in the three regions: $0 \le \rho \le a$, $a \le \rho \le b$, and $\rho \ge b$.

(c) Find the $\mathbf{B}$ field in the three regions. Check to make sure that your answer is consistent with what you know about solenoids. (Hint: $\mathbf{B} \equiv 0$ outside the solenoid.)

Find solution for $f(\rho)$:

$$f(\rho) = \begin{cases} \frac{\mu_0 J_0}{2}(b-a)\rho & \text{for } 0 \le \rho \le a \\ \frac{\mu_0 J_0}{6}\left(3b\rho - 2\rho^2 - \frac{a^3}{\rho}\right) & \text{for } a \le \rho \le b \\ \frac{\mu_0 J_0}{6}\left(\frac{b^3 - a^3}{\rho}\right) & \text{for } \rho > b \end{cases}$$

Find $\mathbf{B}$: $\quad \mathbf{B} = \nabla \times \mathbf{A} = \left(\frac{1}{\rho}\frac{d(\rho f(\rho))}{d\rho}\right)\hat{\mathbf{z}}$

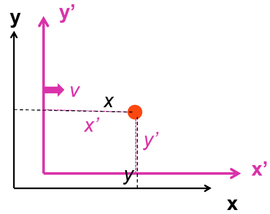
$$\mathbf{B}(\rho) = \begin{cases} \mu_0 J_0 (b-a)\hat{\mathbf{z}} & \text{for } 0 \le \rho \le a \\ \mu_0 J_0 (b-\rho)\hat{\mathbf{z}} & \text{for } a \le \rho \le b \\ 0 & \text{for } \rho > b \end{cases}$$

Basic equations of electrodynamics

CGS (Gaussian)	SI
$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$
$\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
$u = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$	$u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$
$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$	$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$

Notions of special relativity

- The basic laws of physics are the same in all frames of reference (at rest or moving at constant velocity).
- The speed of light in vacuum *c* is the same in all frames of reference.

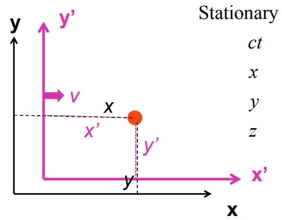


Lorentz transformations

Convenient notation :

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



Stationary frame		Moving frame
ct	$=$	$\gamma(ct'+\beta x')$
x	$=$	$\gamma(x'+\beta ct')$
y	$=$	y'
z	$=$	z'

Lorentz transformations -- continued

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \mathcal{L} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \qquad \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \mathcal{L}^{-1} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Notice :

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

Examples of other 4-vectors applicable to the Lorentz transformation:

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \qquad \text{Note: } \omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$$

$$\begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} = \mathcal{L} \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} \qquad \begin{pmatrix} E' \\ p'_x c \\ p'_y c \\ p'_z c \end{pmatrix} = \mathcal{L}^{1} \begin{pmatrix} E \\ p_x c \\ p_y c \\ p_z c \end{pmatrix} \qquad \text{Note: } E^2 - p^2c^2 = E'^2 - p'^2c^2$$

The Doppler Effect

For the moving frame with $\mathbf{v} = v\hat{\mathbf{x}}$:

$$\mathcal{L} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathcal{L}^{1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \mathcal{L} \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} \qquad \begin{pmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \mathcal{L}^{1} \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} \qquad \text{Note: } \omega t - \mathbf{k}\cdot\mathbf{r} = \omega' t' - \mathbf{k}'\cdot\mathbf{r}'$$

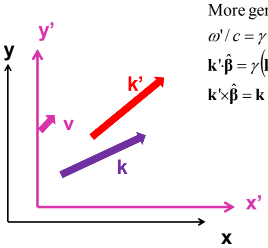
$$\omega'/c = \gamma(\omega/c - \beta k_x) \qquad k'_x = \gamma(k_x - \beta\omega/c)$$

$$k'_y = k_y \qquad k'_z = k_z$$

The Doppler Effect -- continued

$$\omega'/c = \gamma(\omega/c - \beta k_x) \qquad k'_x = \gamma(k_x - \beta\omega/c)$$

$$k'_y = k_y \qquad k'_z = k_z$$



More generally:

$$\omega'/c = \gamma(\omega/c - \boldsymbol{\beta}\cdot\mathbf{k}) \equiv \gamma(\omega/c - \beta k\cos\theta)$$

$$\mathbf{k}'\cdot\hat{\boldsymbol{\beta}} = \gamma(\mathbf{k}\cdot\hat{\boldsymbol{\beta}} - \beta\omega/c) \equiv k'\cos\theta' = \gamma(k\cos\theta - \beta\omega/c)$$

$$\mathbf{k}'\times\hat{\boldsymbol{\beta}} = \mathbf{k}\times\hat{\boldsymbol{\beta}}$$

For $\theta = 0$: $(k = \omega/c)$

$$\omega' = \omega\gamma(1-\beta) \qquad \Rightarrow \omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}}$$

For $\theta \neq 0$: $(k = \omega/c)$

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

Electromagnetic Doppler Effect (θ=0)

$$\omega' = \omega\sqrt{\frac{1-\beta}{1+\beta}} \qquad \beta \equiv \frac{v_{\text{source}} - v_{\text{detector}}}{c}$$

Sound Doppler Effect (θ=0)

$$\omega' = \omega\left(\frac{1 \pm v_{\text{detector}}/c_s}{1 \mp v_{\text{source}}/c_s}\right)$$

Lorentz transformation of the velocity

Stationary frame		Moving frame
ct	=	$\gamma(ct' + \beta x')$
x	=	$\gamma(x' + \beta ct')$
y	=	y'
z	=	z'

For an infinitesimal increment:

Stationary frame		Moving frame
cdt	=	$\gamma(cdt' + \beta dx')$
dx	=	$\gamma(dx' + \beta cdt')$
dy	=	dy'
dz	=	dz'

Lorentz transformation of the velocity -- continued

Stationary frame		Moving frame
cdt	=	$\gamma(cdt' + \beta dx')$
dx	=	$\gamma(dx' + \beta cdt')$
dy	=	dy'
dz	=	dz'

Define: $u_x \equiv \frac{dx}{dt} \quad u_y \equiv \frac{dy}{dt} \quad u_z \equiv \frac{dz}{dt}$

$u'_x \equiv \frac{dx'}{dt'} \quad u'_y \equiv \frac{dy'}{dt'} \quad u'_z \equiv \frac{dz'}{dt'}$

$$\frac{dx}{dt} = \frac{\gamma(dx' + \beta cdt')}{\gamma(dt' + \beta dx'/c)} = \frac{u'_x + v}{1 + vu'_x/c^2} = u_x$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \beta dx'/c)} = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} = u_y$$

Example of velocity variation with β: