PHY 712 Electrodynamics
10-10:50 AM MWF Olin 107

Plan for Lecture 18:

Finish reading Chap. 7; start Chap. 8

A. Summary of results for plane waves

B. Electromagnetic waves in an ideal conductor

C. TEM electromagnetic modes

8	Mon: 02/03/2014	Chap. 4	Multipole analysis of charge distributions	#8	02/05/2014
9	Wed: 02/05/2014	Chap. 4	Dipoles and dielectrics	#9	02/07/2014
10	Fri: 02/07/2014	Chap. 4	Dipoles and dielectrics	#10	02/10/2014
11	Mon: 02/10/2014	Chap. 5	Magnetostatics	#11	02/12/2014
12	Wed: 02/12/2014	Chap. 5	Magnetostatics	#12	02/14/2014
	Fri: 02/14/2014		Class cancelled because of weather		
13	Mon: 02/17/2014	Chap. 5	Magnetostatics	#13	02/28/2014
14	Mon: 02/17/2014	Chap. 6	Maxwell's equations	#14	02/28/2014
15	Wed: 02/19/2014	Chap. 6	Electromagnetic energy and force	#15	02/28/2014
16	Fri: 02/21/2014	Chap. 7	Electromagnetic plane waves	#16	02/28/2014
17	Fri: 02/21/2014	Chap. 7	Dynamic dielectric media and their effects	#17	02/28/2014
	Mon: 02/24/2014		No class -- NAWH out of town		
	Wed: 02/26/2014		No class -- NAWH out of town		
18	Fri: 02/28/2014	Chap. 7	Complex dielectrics; TEM modes		
	Mon: 03/03/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Wed: 03/05/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Fri: 03/07/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Mon: 03/10/2014	*Spring Break*			
	Wed: 03/12/2014	*Spring Break*			
	Fri: 03/14/2014	*Spring Break*			
	Mon: 03/17/2014		Take-home exam due		

Review: Electromagnetic plane waves in isotropic medium with real permeability and permittivity: μ ε.

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

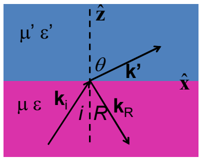
Poynting vector for plane electromagnetic waves:

$$\langle \mathbf{S}\rangle_{avg} = \frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\hat{\mathbf{k}}$$

Energy density for plane electromagnetic waves:

$$\langle u\rangle_{avg} = \frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$

Reflection and refraction between two isotropic media

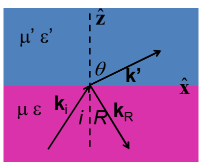


Reflectance, transmittance:

$$\mathrm{R} = \frac{\mathbf{S}_R\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 \qquad \mathrm{T} = \frac{\mathbf{S}'\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}} = \left|\frac{E'_0}{E_{0i}}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'}\frac{\cos\theta}{\cos i}$$

Note that $\mathrm{R}+\mathrm{T}=1$

Reflection and refraction between two isotropic media -- continued



For each wave:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct\right)}\right) \qquad n^2 = c^2\mu\varepsilon$$

$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

Matching condition at interface:

$$n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$$

If $n > n'$, for $i > i_0 \equiv \sin^{-1}\left(\frac{n'}{n}\right)$,

refracted field no longer propagates in medium $\mu'\varepsilon'$

Total internal reflection:

$$n'\cos\theta = i\sqrt{n^2\sin^2 i - n'^2} = i\,n\sqrt{\frac{\sin^2 i}{\sin^2 i_0}-1}$$

$$\mathbf{E}'(\mathbf{r},t) = e^{-\left(\frac{n\omega}{c}\sqrt{\frac{\sin^2 i}{\sin^2 i_0}-1}\right)z}\,\Re\left(\mathbf{E}'_0\, e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_\parallel\cdot\mathbf{r}-ct\right)}\right)$$

For s-polarization

$$\frac{E_{0R}}{E_{0i}} = \frac{n\cos i - \frac{\mu}{\mu'}n'\cos\theta}{n\cos i + \frac{\mu}{\mu'}n'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}n'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

For p-polarization

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'}n'\cos i - n\cos\theta}{\frac{\mu}{\mu'}n'\cos i + n\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{\frac{\mu}{\mu'}n'\cos i + n\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

Special case: normal incidence (i=0, θ=0)

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n} \qquad \frac{E'_0}{E_{0i}} = \frac{2n}{\frac{\mu}{\mu'}n' + n}$$

Reflectance, transmittance:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2$$

$$\mathrm{T} = \left|\frac{E'_0}{E_{0i}}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'} = \left|\frac{2n}{\frac{\mu}{\mu'}n' + n}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'}$$

Extension to complex refractive index $n = n_R + i\, n_I$

Suppose $\mu = \mu'$, n = real, $n' = n'_R + i n'_I$

Reflectance at normal incidence:

$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'}n' - n}{\frac{\mu}{\mu'}n' + n}\right|^2 = \frac{(n'_R - n)^2 + (n'_I)^2}{(n'_R + n)^2 + (n'_I)^2}$$

Note that for $n'_I >> |n'_R \pm n|$:

$$\mathrm{R} \approx 1$$

Fields near the surface on an ideal conductor

Suppose for an isotropic medium: $\mathbf{D} = \varepsilon_b \mathbf{E} \qquad \mathbf{J} = \sigma \mathbf{E}$

Maxwell's equations in terms of $\mathbf{H}$ and $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon_b \frac{\partial \mathbf{E}}{\partial t}$$

$$\left(\nabla^2 - \mu\sigma \frac{\partial}{\partial t} - \mu\varepsilon_b \frac{\partial^2}{\partial t^2}\right)\mathbf{F} = 0 \qquad \mathbf{F} = \mathbf{E}, \mathbf{H}$$

Plane wave form for $\mathbf{E}$:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \text{where } \mathbf{k} = (n_R + i n_I)\frac{\omega}{c}\hat{\mathbf{k}}$$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i n_R (\omega/c)\hat{\mathbf{k}}\cdot\mathbf{r} - i\omega t}\right)$$

Fields near the surface on an ideal conductor -- continued

For our system:

$$\frac{\omega}{c} n_R = \omega \sqrt{\frac{\mu \varepsilon_b}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\omega \varepsilon_b} \right)^2} + 1 \right)^{1/2}$$

$$\frac{\omega}{c} n_I = \omega \sqrt{\frac{\mu \varepsilon_b}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\omega \varepsilon_b} \right)^2} - 1 \right)^{1/2}$$

For $\frac{\sigma}{\omega} >> 1 \quad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu \sigma \omega}{2}} \equiv \frac{1}{\delta}$

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r}/\delta} \Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r}/\delta - i\omega t} \right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta \mu \omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t)$$

Fields near the surface on an ideal conductor -- continued

For $\frac{\sigma}{\omega} >> 1 \quad \frac{\omega}{c} n_R \approx \frac{\omega}{c} n_I \approx \sqrt{\frac{\mu \sigma \omega}{2}} \equiv \frac{1}{\delta}$

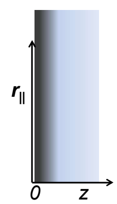
In this limit, $\sqrt{\frac{\mu \varepsilon}{\mu_0 \varepsilon_0}} = c\sqrt{\mu \varepsilon} = n_R + i n_I = \frac{c}{\omega} \frac{1}{\delta} (1+i)$

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r}/\delta} \Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r}/\delta - i\omega t} \right)$$

$$\mathbf{D}(\mathbf{r},t) = \varepsilon \mathbf{E}(\mathbf{r},t) = \frac{i\sigma}{\omega} \mathbf{E}(\mathbf{r},t)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta \mu \omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t) = \mu \mathbf{H}(\mathbf{r},t) = \frac{n}{c} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta \omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t)$$



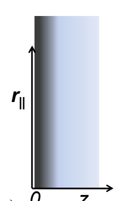
Fields near the surface on an ideal conductor -- continued

$$\mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r}/\delta} \Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r}/\delta - i\omega t} \right)$$

$$\mathbf{D}(\mathbf{r},t) = \varepsilon \mathbf{E}(\mathbf{r},t) = \frac{i\sigma}{\omega} \mathbf{E}(\mathbf{r},t)$$

$$\mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta \mu \omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t) = \mu \mathbf{H}(\mathbf{r},t) = \frac{n}{c} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t) = \frac{1+i}{\delta \omega} \hat{\mathbf{k}} \times \mathbf{E}(\mathbf{r},t)$$



Note that the $\mathbf{H}$ field is larger than $\mathbf{E}$ field so we can write:

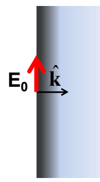
$$\mathbf{H}(\mathbf{r},t) = e^{-\hat{\mathbf{k}} \cdot \mathbf{r}/\delta} \Re\left(\mathbf{H}_0 e^{i\hat{\mathbf{k}} \cdot \mathbf{r}/\delta - i\omega t} \right)$$

$$\mathbf{E}(\mathbf{r},t) = \delta \mu \omega \frac{1-i}{2} \hat{\mathbf{k}} \times \mathbf{H}(\mathbf{r},t)$$

Boundary values for ideal conductor

$$\Rightarrow \mathbf{E}(\mathbf{r},t) = e^{-\hat{\mathbf{k}}\cdot\mathbf{r}/\delta}\,\Re\left(\mathbf{E}_0 e^{i\hat{\mathbf{k}}\cdot\mathbf{r}/\delta - i\omega t}\right)$$

$$\Rightarrow \mathbf{H}(\mathbf{r},t) = \frac{n}{c\mu}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \Re\left(\frac{1+i}{\delta\mu\omega}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)\right)$$



At the boundary of an ideal conductor, the **E** and **H** fields decay in the direction normal to the interface, the field directions are in the plane of the interface.

Waveguide terminology

- TEM: transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)
- TM: transverse magnetic (H field is perpendicular to wave propagation direction)
- TE: transverse electric (E field is perpendicular to wave propagation direction)

TEM waves

Transverse electric and magnetic (both E and H fields are perpendicular to wave propagation direction)

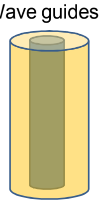
In the free space or within a non-conducting medium; the "normal" electromagnetic modes are TEM:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}\cdot\mathbf{r} - ct\right)}\right) \qquad n^2 = c^2\mu\varepsilon$$

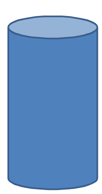
$$\mathbf{B}(\mathbf{r},t) = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t) = \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t)$$

$$\hat{\mathbf{k}}\cdot\mathbf{E} = 0 = \hat{\mathbf{k}}\cdot\mathbf{B}$$

Wave guides

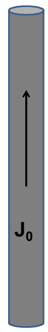


Coaxial cable
TEM modes



Simple optical pipe
TE or TM modes

Comment on HW #11



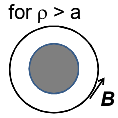
1. Consider an infinitely long wire with radius a, oriented along the **z** axis. There is a steady uniform current inside the wire. Specifically the current is along the z-axis with the magnitude of J_0 for $\rho \le a$ and zero for $\rho > a$, where ρ denotes the radial parameter of the natural cylindrical coordinates of the system.
   a. Find the vector potential (**A**) for all ρ.
   b. Find the magnetic flux field (**B**) for all ρ.

Solution to problem using PHY 114 ideas

In this case, it is convenient to solve part b first.

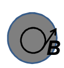
Top view for $\rho < a$

Top view for $\rho > a$



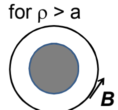
Comment on HW #11 -- continued

Top view for $\rho < a$



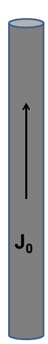
$$\oint \mathbf{B}\cdot \mathbf{d}\ell = \mu_0 \int \mathbf{J}\cdot \mathbf{dA}$$
$$2\pi\rho B = \mu_0 J_0 \pi \rho^2$$
$$B = \frac{\mu_0 J_0 \rho}{2}$$
$$\mathbf{B} = \frac{\mu_0 J_0 \rho}{2}\hat{\boldsymbol{\varphi}} = \nabla \times \mathbf{A}$$
$$\mathbf{A} = -\frac{\mu_0 J_0 \left(\rho^2 - a^2\right)}{4}\hat{\mathbf{z}}$$

Top view for $\rho > a$



$$\oint \mathbf{B}\cdot \mathbf{d}\ell = \mu_0 \int \mathbf{J}\cdot \mathbf{dA}$$
$$2\pi\rho B = \mu_0 J_0 \pi a^2$$
$$B = \frac{\mu_0 J_0 a^2}{2\rho}$$
$$\mathbf{B} = \frac{\mu_0 J_0 a^2}{2\rho}\hat{\boldsymbol{\varphi}} = \nabla \times \mathbf{A}$$
$$\mathbf{A} = -\frac{\mu_0 J_0 a^2 \ln\left(\rho / a\right)}{2}\hat{\mathbf{z}}$$

Comment on HW #11 -- continued



Alternative treatment using differential equations:

$$-\nabla^2 \mathbf{A} = \begin{cases} \mu_0 J_0 \hat{\mathbf{z}} & \text{for } \rho \le a \\ 0 & \text{for } \rho > a \end{cases}$$

$$-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho \frac{\partial A_z(\rho)}{\partial \rho} = \begin{cases} \mu_0 J_0 & \text{for } \rho \le a \\ 0 & \text{for } \rho > a \end{cases}$$

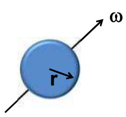
$$A_z(\rho) = \begin{cases} -\frac{\mu_0 J_0 \rho^2}{4} + C_1 & \text{for } \rho \le a \\ C_2 + C_3 \ln(\rho) & \text{for } \rho > a \end{cases}$$

Choosing constants from continuity requirements:

$$A_z(\rho) = \begin{cases} -\frac{\mu_0 J_0 \rho^2}{4} + \frac{\mu_0 J_0 a^2}{4} & \text{for } \rho \le a \\ -\frac{\mu_0 J_0 a^2}{2}\ln(\rho / a) & \text{for } \rho > a \end{cases}$$

$$\mathbf{B} = -\frac{\partial A_z(\rho)}{\partial \rho}\hat{\boldsymbol{\varphi}}$$

Comment on HW #12



A sphere of radius a carries a uniform surface charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential **A** and magnetic field **B** both inside and outside the sphere.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

$$\mathbf{J}(\mathbf{r}') = \begin{cases} \sigma\ \delta(r'-a)\ \boldsymbol{\omega}\times\mathbf{r}' & \text{for } r' \leq a \\ 0 & \text{otherwise} \end{cases}$$

Note that: $$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

and: $$\int d\Omega' \sum_m Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')\,\mathbf{r}' = \frac{r'}{r}\mathbf{r}\,\delta_{l1}.$$

Comment on HW #12 -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0\sigma}{4\pi}\frac{\boldsymbol{\omega}\times\mathbf{r}}{r}\frac{4\pi}{3}\int_0^a r'^3 dr'\,\delta(r'-a)\frac{r_<}{r_>^2}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0\sigma}{3}\boldsymbol{\omega}\times\mathbf{r}\begin{cases} a & \text{for } r \leq a \\ \frac{a^4}{r^3} & \text{for } r > a \end{cases}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0\sigma}{3}\begin{cases} 2\boldsymbol{\omega}a & \text{for } r \leq a \\ \frac{a^4}{r^3}\left(3\left(\hat{\mathbf{r}}\cdot\boldsymbol{\omega}\right)\hat{\mathbf{r}}-\boldsymbol{\omega}\right) & \text{for } r > a \end{cases}$$