PHY 712 Electrodynamics
9-9:50 & 10-10:50 AM Olin 107

Plan for Lecture 16-17:

Read Chapter 7

1. **Plane polarized electromagnetic waves**
2. **Reflectance and transmittance of electromagnetic waves – extension to anisotropy and complexity**
3. **Frequency dependence of dielectric materials; Drude model**
4. **Kramers-Kronig relationships**

	Wed: 01/29/2014		NAWH out of town - no class		
7	Fri: 01/31/2014	Chap. 3	Cylindrical and spherical geometries	#7	02/05/2014
8	Mon: 02/03/2014	Chap. 4	Multipole analysis of charge distributions	#8	02/05/2014
9	Wed: 02/05/2014	Chap. 4	Dipoles and dielectrics	#9	02/07/2014
10	Fri: 02/07/2014	Chap. 4	Dipoles and dielectrics	#10	02/10/2014
11	Mon: 02/10/2014	Chap. 5	Magnetostatics	#11	02/12/2014
12	Wed: 02/12/2014	Chap. 5	Magnetostatics	#12	02/14/2014
	Fri: 02/14/2014		Class cancelled because of weather		
13	Mon: 02/17/2014	Chap. 5	Magnetostatics	#13	02/19/2014
14	Mon: 02/17/2014	Chap. 6	Maxwell's equations	#14	02/19/2014
15	Wed: 02/19/2014	Chap. 6	Electromagnetic energy and force	#15	02/21/2014
16	Fri: 02/21/2014	Chap. 7	Electromagnetic plane waves	#16	02/28/2014
17	Fri: 02/21/2014	Chap. 7	Dynamic dielectric media and their effects	#17	02/28/2014
	Mon: 02/24/2014		No class -- NAWH out of town		
	Wed: 02/26/2014		No class -- NAWH out of town		
18	Fri: 02/28/2014	Chap. 7	Dynamic dielectric media and their effects		
	Mon: 03/03/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Wed: 03/05/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Fri: 03/07/2014	*APS Meeting*	Take-home exam (no class meeting)		
	Mon: 03/10/2014	*Spring Break*			
	Wed: 03/12/2014	*Spring Break*			
	Fri: 03/14/2014	*Spring Break*			
	Mon: 03/17/2014		Take-home exam due		

Maxwell's equations

For linear isotropic media and no sources: $\mathbf{D}=\varepsilon\mathbf{E};\quad \mathbf{B}=\mu\mathbf{H}$

Coulomb's law: $\nabla\cdot\mathbf{E}=0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}=0$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

Analysis of Maxwell's equations without sources -- continued:

$$\text{Coulomb's law:} \quad \nabla \cdot \mathbf{E} = 0$$

$$\text{Ampere-Maxwell's law:} \quad \nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} = 0$$

$$\text{Faraday's law:} \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\text{No magnetic monopoles:} \quad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{B} - \mu\varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial (\nabla \times \mathbf{E})}{\partial t}$$

$$= -\nabla^2 \mathbf{B} + \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla \times \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) = -\nabla^2 \mathbf{E} + \frac{\partial (\nabla \times \mathbf{B})}{\partial t}$$

$$= -\nabla^2 \mathbf{E} + \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Analysis of Maxwell's equations without sources -- continued:
Both E and B fields are solutions to a wave equation:

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{where} \quad v^2 \equiv c^2 \frac{\mu_0 \varepsilon_0}{\mu\varepsilon} \equiv \frac{c^2}{n^2}$$

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

Analysis of Maxwell's equations without sources -- continued:

Plane wave solutions to wave equation:

$$\mathbf{B}(\mathbf{r},t) = \Re\left(\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right) \qquad \mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}\right)$$

$$|\mathbf{k}|^2 = \left(\frac{\omega}{v}\right)^2 = \left(\frac{n\omega}{c}\right)^2 \qquad \text{where } n \equiv \sqrt{\frac{\mu\varepsilon}{\mu_0 \varepsilon_0}}$$

Note: ε, μ, n, k can all be complex; for the moment we will assume that they are all real (no dissipation).

Note that $\mathbf{E}_0$ and $\mathbf{B}_0$ are not independent;

$$\text{from Faraday's law:} \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\Rightarrow \mathbf{B}_0 = \frac{\mathbf{k} \times \mathbf{E}_0}{\omega} = \frac{n\hat{\mathbf{k}} \times \mathbf{E}_0}{c}$$

$$\text{also note:} \quad \hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0 \quad \text{and} \quad \hat{\mathbf{k}} \cdot \mathbf{B}_0 = 0$$

Analysis of Maxwell's equations without sources -- continued:

Summary of plane electromagnetic waves:

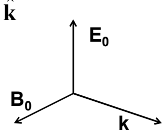
$$\mathbf{B}(\mathbf{r},t)=\Re\left(\frac{n\hat{\mathbf{k}}\times\mathbf{E}_0}{c}e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)\qquad \mathbf{E}(\mathbf{r},t)=\Re\left(\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}\right)$$

$$|\mathbf{k}|^2=\left(\frac{\omega}{v}\right)^2=\left(\frac{n\omega}{c}\right)^2 \quad \text{where } n\equiv\sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \quad \text{and } \hat{\mathbf{k}}\cdot\mathbf{E}_0=0$$

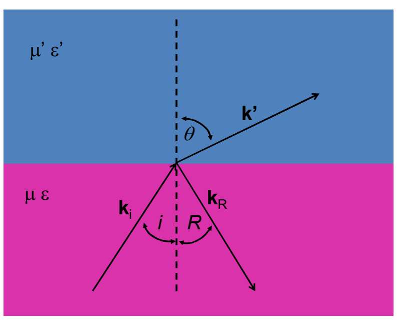
Poynting vector and energy density:

$$\langle \mathbf{S}\rangle_{avg}=\frac{n|\mathbf{E}_0|^2}{2\mu c}\hat{\mathbf{k}}=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}_0|^2\,\hat{\mathbf{k}}$$

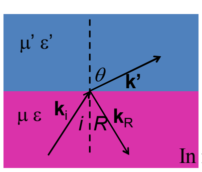
$$\langle u\rangle_{avg}=\frac{1}{2}\varepsilon|\mathbf{E}_0|^2$$



Reflection and refraction of plane electromagnetic waves at a plane interface between dielectrics (assumed to be lossless)



Reflection and refraction -- continued



In medium $\mu'\varepsilon'$:

$$\mathbf{E}'(\mathbf{r},t)=\Re\left(\mathbf{E}'_0\, e^{i\frac{\omega}{c}\left(n'\hat{\mathbf{k}}'\cdot\mathbf{r}-ct\right)}\right)$$

$$\mathbf{B}'(\mathbf{r},t)=\frac{n'}{c}\hat{\mathbf{k}}'\times\mathbf{E}'(\mathbf{r},t)=\sqrt{\mu'\varepsilon'}\hat{\mathbf{k}}'\times\mathbf{E}(\mathbf{r},t)$$

In medium $\mu\varepsilon$:

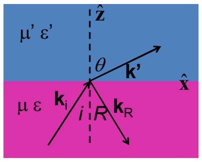
$$\mathbf{E}_i(\mathbf{r},t)=\Re\left(\mathbf{E}_{0i}e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_i\cdot\mathbf{r}-ct\right)}\right)$$

$$\mathbf{B}_i(\mathbf{r},t)=\frac{n}{c}\hat{\mathbf{k}}_i\times\mathbf{E}_i(\mathbf{r},t)=\sqrt{\mu\varepsilon}\hat{\mathbf{k}}_i\times\mathbf{E}_i(\mathbf{r},t)$$

$$\mathbf{E}_R(\mathbf{r},t)=\Re\left(\mathbf{E}_{0R}e^{i\frac{\omega}{c}\left(n\hat{\mathbf{k}}_R\cdot\mathbf{r}-ct\right)}\right)$$

$$\mathbf{B}_R(\mathbf{r},t)=\frac{n}{c}\hat{\mathbf{k}}_R\times\mathbf{E}_R(\mathbf{r},t)=\sqrt{\mu\varepsilon}\hat{\mathbf{k}}_R\times\mathbf{E}_R(\mathbf{r},t)$$

Reflection and refraction -- continued



Snell's law -- matching phase factors at boundary plane:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

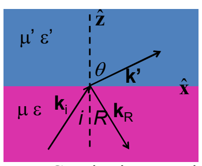
$$\hat{\mathbf{k}}' \cdot \mathbf{r} = x \sin\theta$$

$$\hat{\mathbf{k}}_i \cdot \mathbf{r} = x \sin i = \hat{\mathbf{k}}_R \cdot \mathbf{r} \quad \Rightarrow i = R$$

$$n' \hat{\mathbf{k}}' \cdot \mathbf{r} = n \hat{\mathbf{k}}_i \cdot \mathbf{r} \quad \Rightarrow n' x \sin\theta = n x \sin i$$

Snell's law: $n' \sin\theta = n \sin i$

Reflection and refraction -- continued



Continuity equations at boundary with no sources:

$$\nabla \cdot \mathbf{D} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

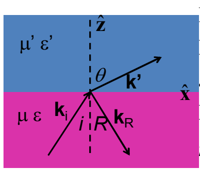
$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0 \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

Matching field amplitudes at boundary plane:

$\mathbf{D} \cdot \hat{\mathbf{z}}$, $\mathbf{B} \cdot \hat{\mathbf{z}}$ continuous

$\mathbf{H} \times \hat{\mathbf{z}}$, $\mathbf{E} \times \hat{\mathbf{z}}$ continuous

Reflection and refraction -- continued



Matching field amplitudes at boundary plane:

$\mathbf{D} \cdot \hat{\mathbf{z}}$ continuous:

$$\varepsilon\left(\mathbf{E}_{0i} + \mathbf{E}_{0R}\right) \cdot \hat{\mathbf{z}} = \varepsilon' \mathbf{E}'_0 \cdot \hat{\mathbf{z}}$$

$\mathbf{B} \cdot \hat{\mathbf{z}}$ continuous:

$$n\left(\hat{\mathbf{k}}_i \times \mathbf{E}_{0i} + \hat{\mathbf{k}}_R \times \mathbf{E}_{0R}\right) \cdot \hat{\mathbf{z}} = n' \hat{\mathbf{k}}' \times \mathbf{E}'_{0i} \cdot \hat{\mathbf{z}}$$

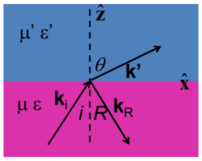
$\mathbf{E} \times \hat{\mathbf{z}}$ continuous:

$$\left(\mathbf{E}_{0i} + \mathbf{E}_{0R}\right) \times \hat{\mathbf{z}} = \mathbf{E}'_0 \times \hat{\mathbf{z}}$$

$\mathbf{H} \times \hat{\mathbf{z}}$ continuous:

$$\frac{n}{\mu}\left(\hat{\mathbf{k}}_i \times \mathbf{E}_{0i} + \hat{\mathbf{k}}_R \times \mathbf{E}_{0R}\right) \times \hat{\mathbf{z}} = \frac{n'}{\mu'} \hat{\mathbf{k}}' \times \mathbf{E}'_{0i} \times \hat{\mathbf{z}}$$

Reflection and refraction -- continued



s-polarization – **E** field "polarized" perpendicular to plane of incidence

$\mathbf{E}\times\hat{\mathbf{z}}$ continuous:

$$(\mathbf{E}_{0i}+\mathbf{E}_{0R})\times\hat{\mathbf{z}}=\mathbf{E}'_0\times\hat{\mathbf{z}}$$

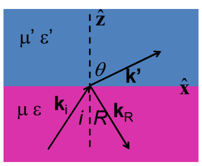
$\mathbf{H}\times\hat{\mathbf{z}}$ continuous:

$$\frac{n}{\mu}\left(\hat{\mathbf{k}}_i\times\mathbf{E}_{0i}+\hat{\mathbf{k}}_R\times\mathbf{E}_{0R}\right)\times\hat{\mathbf{z}}=\frac{n'}{\mu'}\hat{\mathbf{k}}'\times\mathbf{E}'_{0i}\times\hat{\mathbf{z}}$$

$$\frac{E_{0R}}{E_{0i}}=\frac{n\cos i-\frac{\mu}{\mu'}n'\cos\theta}{n\cos i+\frac{\mu}{\mu'}n'\cos\theta}\qquad\frac{E'_0}{E_{0i}}=\frac{2n\cos i}{n\cos i+\frac{\mu}{\mu'}n'\cos\theta}$$

Note that: $n'\cos\theta=\sqrt{n'^2-n^2\sin^2 i}$

Reflection and refraction -- continued



p-polarization – **E** field "polarized" parallel to plane of incidence

$\mathbf{D}\cdot\hat{\mathbf{z}}$ continuous:

$$\varepsilon(\mathbf{E}_{0i}+\mathbf{E}_{0R})\cdot\hat{\mathbf{z}}=\varepsilon'\mathbf{E}'_0\cdot\hat{\mathbf{z}}$$

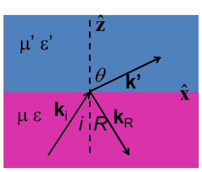
$\mathbf{H}\times\hat{\mathbf{z}}$ continuous:

$$\frac{n}{\mu}\left(\hat{\mathbf{k}}_i\times\mathbf{E}_{0i}+\hat{\mathbf{k}}_R\times\mathbf{E}_{0R}\right)\times\hat{\mathbf{z}}=\frac{n'}{\mu'}\hat{\mathbf{k}}'\times\mathbf{E}'_{0i}\times\hat{\mathbf{z}}$$

$$\frac{E_{0R}}{E_{0i}}=\frac{\frac{\mu}{\mu'}n'\cos i-n\cos\theta}{\frac{\mu}{\mu'}n'\cos i+n\cos\theta}\qquad\frac{E'_0}{E_{0i}}=\frac{2n\cos i}{\frac{\mu}{\mu'}n'\cos i+n\cos\theta}$$

Note that: $n'\cos\theta=\sqrt{n'^2-n^2\sin^2 i}$

Reflection and refraction -- continued



Reflectance, transmittance:

$$\mathrm{R}=\frac{\mathbf{S}_R\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E_{0R}}{E_{0i}}\right|^2\qquad\mathrm{T}=\frac{\mathbf{S}'\cdot\hat{\mathbf{z}}}{\mathbf{S}_i\cdot\hat{\mathbf{z}}}=\left|\frac{E'_0}{E_{0i}}\right|^2\frac{n'}{n}\frac{\mu}{\mu'}\frac{\cos\theta}{\cos i}$$

Note that $\mathrm{R}+\mathrm{T}=1$

For s-polarization

$$\frac{E_{0R}}{E_{0i}} = \frac{n\cos i - \frac{\mu}{\mu'} n'\cos\theta}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'} n'\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

For p-polarization

$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'} n'\cos i - n\cos\theta}{\frac{\mu}{\mu'} n'\cos i + n\cos\theta} \qquad \frac{E'_0}{E_{0i}} = \frac{2n\cos i}{\frac{\mu}{\mu'} n'\cos i + n\cos\theta}$$

Note that: $n'\cos\theta = \sqrt{n'^2 - n^2\sin^2 i}$

Special case: normal incidence (i=0, θ=0)

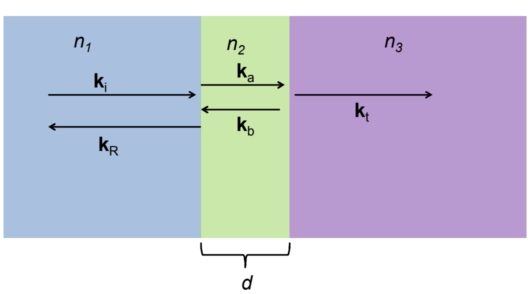
$$\frac{E_{0R}}{E_{0i}} = \frac{\frac{\mu}{\mu'} n' - n}{\frac{\mu}{\mu'} n' + n} \qquad \frac{E'_0}{E_{0i}} = \frac{2n}{\frac{\mu}{\mu'} n' + n}$$

Reflectance, transmittance:

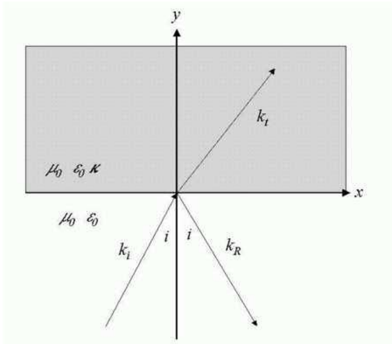
$$\mathrm{R} = \left|\frac{E_{0R}}{E_{0i}}\right|^2 = \left|\frac{\frac{\mu}{\mu'} n' - n}{\frac{\mu}{\mu'} n' + n}\right|^2$$

$$\mathrm{T} = \left|\frac{E'_0}{E_{0i}}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'} = \left|\frac{2n}{\frac{\mu}{\mu'} n' + n}\right|^2 \frac{n'}{n}\frac{\mu}{\mu'}$$

Multilayer dielectrics (Problem #7.2)



Extension of analysis to anisotropic media --



Consider the problem of determining the reflectance from an anisotropic medium with isotropic permeability μ_0 and anisotropic permittivity $\varepsilon_0 \boldsymbol{\kappa}$ where:

$$\boldsymbol{\kappa} \equiv \begin{pmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}$$

By assumption, the wave vector in the medium is confined to the *x-y* plane and will be denoted by

$$\mathbf{k}_t \equiv \frac{\omega}{c}(n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}}), \text{ where } n_x \text{ and } n_y \text{ are to be determined.}$$

The electric field inside the medium is given by:

$$\mathbf{E} = (E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}})\mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

Inside the anisotropic medium, Maxwell's equations are:

$$\nabla \cdot \mathbf{H} = 0 \qquad \nabla \cdot \boldsymbol{\kappa} \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} - i\omega\mu_0\mathbf{H} = 0 \qquad \nabla \times \mathbf{H} + i\omega\epsilon_0\boldsymbol{\kappa} \cdot \mathbf{E} = 0$$

After some algebra, the equation for **E** is:

$$\begin{pmatrix} \kappa_{xx} - n_y^2 & n_x n_y & 0 \\ n_x n_y & \kappa_{yy} - n_x^2 & 0 \\ 0 & 0 & \kappa_{zz} - (n_x^2 + n_y^2) \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0.$$

From **E, H** can be determined from

$$\mathbf{H} = \frac{1}{\mu_0 c}\left\{E_z(n_y\hat{\mathbf{x}} - n_x\hat{\mathbf{y}}) + (E_y n_x - E_x n_y)\hat{\mathbf{z}}\right\}\mathrm{e}^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

The fields for the incident and reflected waves are the same as for the isotropic case.

$$\mathbf{k}_i = \frac{\omega}{c}(\sin i\hat{\mathbf{x}} + \cos i\hat{\mathbf{y}}),$$

$$\mathbf{k}_R = \frac{\omega}{c}(\sin i\hat{\mathbf{x}} - \cos i\hat{\mathbf{y}}).$$

Note that, consistent with Snell's law: $n_x = \sin i$
Continuity conditions at the *y=0* plane must be applied for the following fields:

$$\mathbf{H}(x,0,z,t),\ E_x(x,0,z,t),\ E_z(x,0,z,t),\ \text{and}\ D_y(x,0,z,t).$$

There will be two different solutions, depending of the polarization of the incident field.

Solution for s-polarization

$$E_x = E_y = 0 \qquad \Rightarrow n_y^2 = \kappa_{zz} - n_x^2$$

$$\mathbf{E} = E_z\hat{\mathbf{z}}e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t} \qquad \mathbf{H} = \frac{1}{\mu_0 c}\left\{E_z(n_y\hat{\mathbf{x}} - n_x\hat{\mathbf{y}})\right\}e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}$$

E_z must be determined from the continuity conditions:

$$E_0 + E_0'' = E_z \qquad (E_0 - E_0'')\cos i = E_z n_y \qquad (E_0 + E_0'')\sin i = E_z n_x$$

$$\frac{E_0''}{E_0} = \frac{\cos i - n_y}{\cos i + n_y}.$$

Solution for p-polarization

$$E_z = 0 \qquad \Rightarrow n_y^2 = \frac{\kappa_{xx}}{\kappa_{yy}}(\kappa_{yy} - n_x^2).$$

$$\mathbf{E} = E_x\left(\hat{\mathbf{x}} - \frac{\kappa_{xx} n_x}{\kappa_{yy} n_y}\hat{\mathbf{y}}\right)e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

$$\mathbf{H} = -\frac{E_x}{\mu_0 c}\frac{\kappa_{xx}}{n_y}\hat{\mathbf{z}}e^{i\frac{\omega}{c}(n_x x + n_y y) - i\omega t}.$$

E_x must be determined from the continuity conditions:

$$(E_0 - E_0'')\cos i = E_x \qquad (E_0 + E_0'') = \frac{\kappa_{xx}}{n_y}E_x \qquad (E_0 + E_0'')\sin i = \frac{\kappa_{xx} n_x}{n_y}E_x.$$

$$\frac{E_0''}{E_0} = \frac{\kappa_{xx}\cos i - n_y}{\kappa_{xx}\cos i + n_y}.$$

Extension of analysis to complex dielectric functions

For simplicity assume that $\mu = \mu_0$

Suppose the dielectric function is complex :

$$\varepsilon = \varepsilon_R + i\varepsilon_I \qquad \frac{\varepsilon}{\varepsilon_0} = (n_R + i n_I)^2 \equiv \alpha + i\beta$$

$$n_R = \left(\frac{\sqrt{\alpha^2+\beta^2}+\alpha}{2}\right)^{1/2} \qquad n_I = \left(\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{2}\right)^{1/2}$$

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}(n\hat{\mathbf{k}}\cdot\mathbf{r}-ct)}\right) = \Re\left(\mathbf{E}_0 e^{i\frac{\omega}{c}(n_R\hat{\mathbf{k}}\cdot\mathbf{r}-ct)}\right) e^{-\frac{\omega}{c}n_I\hat{\mathbf{k}}\cdot\mathbf{r}}$$

Paul Karl Ludwig Drude 1863-1906

http://photos.aip.org/history/Thumbnails/drude_paul_a1.jpg

Drude model:

Vibrations of charged particles near equilibrium:

$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

For $\quad \delta\mathbf{r} \equiv \delta\mathbf{r}_0 e^{-i\omega t}, \qquad \delta\mathbf{r}_0 = \dfrac{q\mathbf{E}_0}{m}\dfrac{1}{\omega_0^2-\omega^2-i\omega\gamma}$

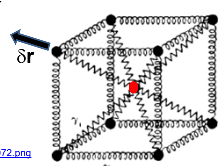
Induced dipole :

$$\mathbf{p} = q\,\delta\mathbf{r} = \frac{q^2\mathbf{E}_0}{m}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}e^{-i\omega t}$$

Displacement field :

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r}-\mathbf{r}_i)$$



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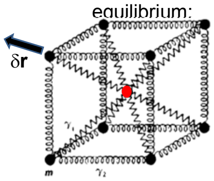
Drude model:

Vibration of particle of charge q and mass m near equilibrium:



http://img.tfd.com/ggse/d6/gsed_0001_0012_0_img2972.png

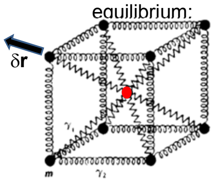
$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\,\mathbf{r} - m\gamma\delta\,\dot{\mathbf{r}}$$

Note that:

- $\gamma > 0$ represents dissipation of energy.
- ω_0 represents the natural frequency of the vibration; $\omega_0=0$ would represent a free (unbound) particle

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\,\mathbf{r} - m\gamma\delta\,\dot{\mathbf{r}}$$

For $\delta\mathbf{r} \equiv \delta\mathbf{r}_0 e^{-i\omega t}$, $\delta\mathbf{r}_0 = \dfrac{q\mathbf{E}_0}{m}\dfrac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$

Induced dipole:

$$\mathbf{p} = q\,\delta\mathbf{r} = \frac{q^2\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}e^{-i\omega t}$$

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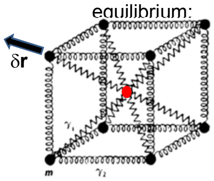

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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\,\mathbf{r} - m\gamma\delta\,\dot{\mathbf{r}}$$

Displacement field:

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

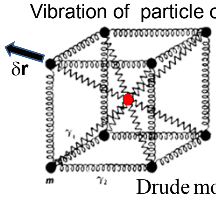
$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i) \approx N\sum_i f_i\,\mathbf{p}_i$$

$N \equiv$ number dipole/volume

$f_i \equiv$ fraction of type i dipoles

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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Drude model expression for permittivity:

$$\mathbf{D} = \varepsilon\mathbf{E} \quad = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\mathbf{E} + N\sum_i f_i\,\mathbf{p}_i$$

$$\mathbf{p}_i \;= q_i\;\delta\mathbf{r} = \frac{q_i^2\mathbf{E}_0}{m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}e^{-i\omega t}$$

$$\varepsilon\mathbf{E} \;= \varepsilon_0\mathbf{E}_0 e^{-i\omega t}\left(1+N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}\right)$$

Drude model dielectric function:

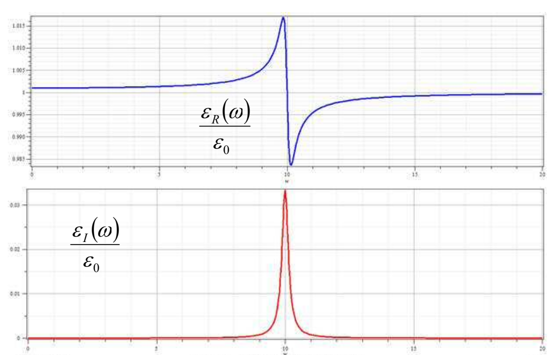
$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1+N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$$

$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0}+i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1+N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{\omega_i^2-\omega^2}{\left(\omega_i^2-\omega^2\right)^2+\omega^2\gamma_i^2}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{\omega\gamma_i}{\left(\omega_i^2-\omega^2\right)^2+\omega^2\gamma_i^2}$$

Drude model dielectric function:



Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$$

For $\omega >> \omega_i$ $\quad \frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1-\frac{1}{\omega^2}\left(N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\right)$

$$\equiv 1-\frac{\omega_P^2}{\omega^2}$$

Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i}$$

For $\omega_0 = 0$ (representing a free particle of charge q_0, mass m_0

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_{i>0} f_i \frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-\omega^2-i\omega\gamma_i} + iNf_0\frac{q_0^2}{\varepsilon_0 m_0}\frac{1}{\omega(\gamma_0-i\omega)}$$

$$\equiv \frac{\varepsilon_b(\omega)}{\varepsilon_0} + i\frac{\sigma(\omega)}{\varepsilon_0\omega}$$

Some details:

$\mathbf{D} = \varepsilon_b\mathbf{E} \qquad \mathbf{J} = \sigma\mathbf{E}$

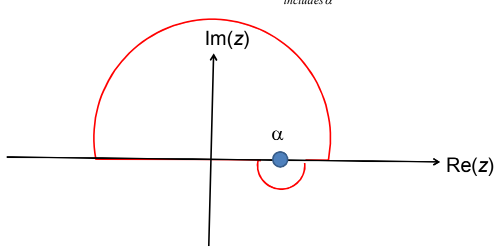
$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} = (\sigma - i\omega\varepsilon_b)\mathbf{E} = \varepsilon\frac{\partial\mathbf{E}}{\partial t} = -i\omega\left(\varepsilon_b + \frac{i\sigma}{\omega}\right)\mathbf{E}$$

$$\Rightarrow \sigma(\omega) = Nf_0\frac{q_0^2}{m_0}\frac{1}{(\gamma_0 - i\omega)}$$

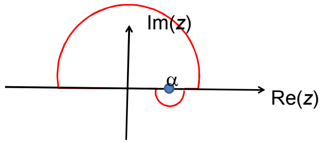
Analytic properties of the dielectric function (in the Drude model or from "first principles" -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function $f(z)$:

$$\oint dz\, f(z) = 0 \qquad f(\alpha) = \frac{1}{2\pi i}\oint_{\text{includes }\alpha} dz\frac{f(z)}{z-\alpha}$$



Kramers-Kronig transform -- continued



$$f(\alpha)=\frac{1}{2\pi i}\oint_{includes\ \alpha} dz\,\frac{f(z)}{z\text{-}\alpha} = \frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R\text{-}\alpha} + \underbrace{\int_{rest} dz\,\frac{f(z)}{z\text{-}\alpha}}_{=0}\right)$$

$$f(\alpha)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R\text{-}\alpha} = \frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R\text{-}\alpha}+\frac{1}{2}f(\alpha)$$

Kramers-Kronig transform -- continued

$$f(\alpha)=\frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R\text{-}\alpha}+\frac{1}{2}f(\alpha)$$

Suppose $f(z_R)=f_R(z_R)+if_I(z_R)$:

$$\Rightarrow \frac{1}{2}\left(f_R(\alpha)+if_I(\alpha)\right)=\frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f_R(z_R)+if_I(z_R)}{z_R\text{-}\alpha}$$

$$\Rightarrow f_R(\alpha)=\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_I(z_R)}{z_R\text{-}\alpha}$$

$$f_I(\alpha)=-\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_R(z_R)}{z_R\text{-}\alpha}$$

Kramers-Kronig transform -- continued

$$f_R(\alpha)=\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_I(z_R)}{z_R\text{-}\alpha}$$

$$f_I(\alpha)=-\frac{1}{\pi}P\int_{-\infty}^{\infty} dz_R\,\frac{f_R(z_R)}{z_R\text{-}\alpha}$$

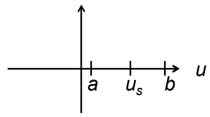
This Kramers-Kronig transform is useful for the dielectric function

when $f(z_R)\Rightarrow\frac{\varepsilon(\omega)}{\varepsilon_0}-1$

Must show that: 1. $f(z)$ is analytic for $z_I\geq 0$

2. $f(z)$ vanishes for $z\rightarrow\infty$

Some practical considerations



Principal parts integration :

$$P\int_a^b du\; g(u) = \lim_{\upsilon\to 0}\left(\int_a^{u_s-\upsilon} du\; g(u) + \int_{u_s+\upsilon}^{b} du\; g(u)\right)$$

Example :

$$P\int_a^b du\; \frac{1}{u\text{-}u_s} = \lim_{\upsilon\to 0}\left(\int_a^{u_s-\upsilon} du\,\frac{1}{u\text{-}u_s} + \int_{u_s+\upsilon}^{b} du\,\frac{1}{u\text{-}u_s}\right)$$

$$= \lim_{\upsilon\to 0}\left(\ln\left(\frac{\upsilon}{u_s-a}\right) + \ln\left(\frac{b-u_s}{\upsilon}\right)\right) = \ln\left(\frac{b-u_s}{u_s-a}\right)$$

More practical considerations

For dielectric function $\varepsilon(\omega)$:

$$\varepsilon(-\omega) = \varepsilon^*(\omega)$$
$$\Rightarrow \varepsilon_R(-\omega) = \varepsilon_R(\omega)$$
$$\Rightarrow \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Analytic properties the dielectric function which justify the treatment of $f(z) \Rightarrow \frac{\varepsilon(z)}{\varepsilon_0} - 1$

Must show that : 1. $f(z)$ is analytic for $z_I \geq 0$

2. $f(z)$ vanishes for $z \to \infty$ (for $z_I \geq 0$)

Analysis for Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let $f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For $|z| >> \omega_i$

$$f(z) \approx -\frac{1}{z^2}\left(N\sum_i f_i \frac{q_i^2}{\varepsilon_0 m_i}\right) \Rightarrow \text{vanishes at large } z$$

Analysis for Drude model dielectric function – continued --
Analytic properties:

$$f(z)=\frac{\varepsilon(z)}{\varepsilon_0}-1=N\sum_i f_i\frac{q_i^2}{\varepsilon_0 m_i}\frac{1}{\omega_i^2-z^2-iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2-z_P{}^2-iz_P\gamma_i=0$

$$z_P=-i\frac{\gamma_i}{2}\pm\sqrt{\omega_i^2-\left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P)\le 0 \Rightarrow f(z)$ is analytic for $\Im(z_P)>0$

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0}-1=\frac{1}{\pi}P\int_{-\infty}^{\infty}d\omega'\frac{\varepsilon_I(\omega')}{\varepsilon_0}\frac{1}{\omega'-\omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0}=-\frac{1}{\pi}P\int_{-\infty}^{\infty}d\omega'\left(\frac{\varepsilon_R(\omega')}{\varepsilon_0}-1\right)\frac{1}{\omega'-\omega}$$

with $\varepsilon_R(-\omega)=\varepsilon_R(\omega)$; $\varepsilon_I(-\omega)=-\varepsilon_I(\omega)$

Further comments on analytic behavior of dielectric function

"Causal" relationship between **E** and **D** fields:

$$\mathbf{D}(\mathbf{r},t)=\varepsilon_0\left\{\mathbf{E}(\mathbf{r},t)+\int_0^{\infty}d\tau\, G(\tau)\mathbf{E}(\mathbf{r},t-\tau)\right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0}-1=\int_0^{\infty}d\tau\, G(\tau)e^{i\omega\tau}$$