PHY 712 Electrodynamics
11-11:50 AM MWF Olin 107

Plan for Lecture 7:

Start reading Chapter 3

Solution of Poisson equation in for special geometries –

A. Cylindrical

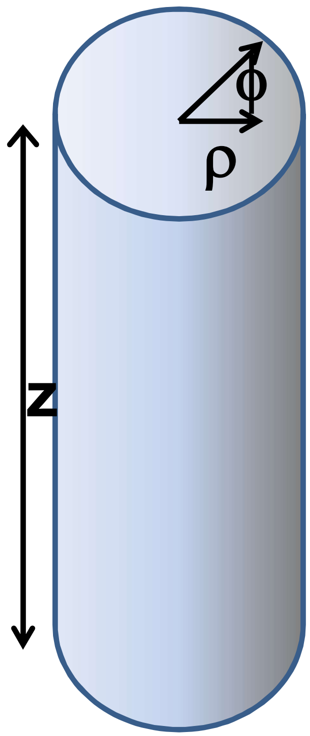
B. Spherical

Course schedule for 2013

(Preliminary schedule -- subject to frequent adjustment.)

Date	JDJ Reading	Topic	Assign.
01-16(Wed)	Chap. 1	Introduction, units and Poisson equation.	#1
01-18(Fri)	Chap. 1	Electronstatic energy calculations	#2
01-21(Mon)	*No class*	*MKL Holiday*	
01-23(Wed)	Chap. 1	Poisson Equation and Green's Functions	#3
01-25(Fri)	Chap. 1 & 2	Green's Theorem and Functions	#4
01-28(Mon)	Chap. 1 & 2	Brief introduction to numerical methods	#5
01-30(Wed)	Chap. 2	Method of images	#6
02-01(Fri)	Chap. 3	Cylindrical and spherical geometries	#7

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence



$$\text{Laplace equation:} \qquad \nabla^2\Phi = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$$

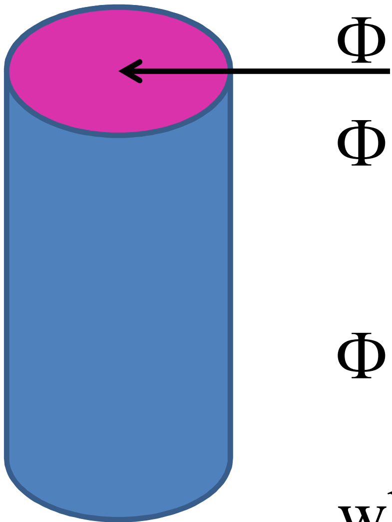
Cylindrical geometry continued:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \qquad \Rightarrow Z(z) = \sinh(kz), \cosh(kz), e^{\pm kz}$$

$$\frac{d^2 Q}{d\phi^2} + m^2 Q = 0 \qquad \Rightarrow Q(\phi) = e^{\pm im\phi}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{m^2}{\rho^2}\right) R = 0 \qquad \Rightarrow J_m(k\rho), N_m(k\rho)$$

Cylindrical geometry example:



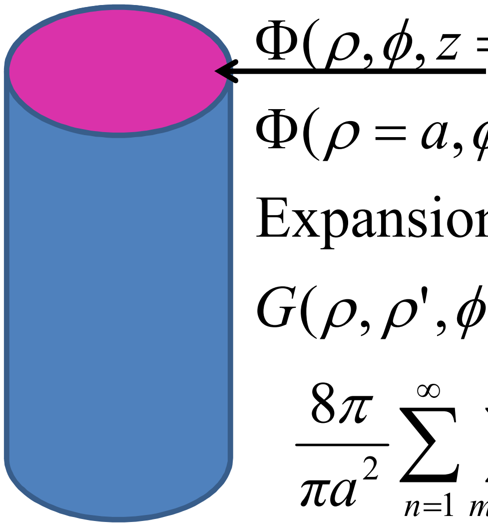
$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin\left(m\phi+\alpha_{mn}\right)$$

$$\text{where} \quad J_m(k_{mn}a)=0$$

Green's function for Dirchelet boundary value inside cylindar:



$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$

$$\Phi(\rho=a,\phi,z)=0,\ \ \Phi(\rho,\phi,z=0)=0$$

Expansion in terms of Bessel function zeros: $J_m(k_{mn}a)=0$

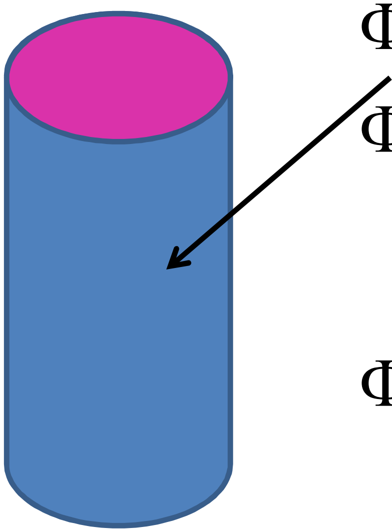
$$G(\rho,\rho',\phi,\phi',z,z')=$$

$$\frac{8\pi}{\pi a^2}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}J_m(k_{mn}\rho)J_m(k_{mn}\rho')\sinh(k_{mn}z_<)\sinh(k_{mn}(L-z_>))}{k_{mn}(J_{m+1}(k_{mn}a))^2\sinh(k_{mn}L)}$$

$$\Phi(\rho,\phi,z)=\frac{1}{4\pi\varepsilon_0}\int_V d\phi'\rho'd\rho'dz'G(\rho,\rho',\phi,\phi',z,z')\rho(\rho',\phi',z')$$

$$+\frac{1}{4\pi}\int_{S;z'=L}d\phi'\rho'd\rho'\left.\frac{\partial G(\rho,\rho',\phi,\phi',z,z')}{\partial z'}\right|_{z'=L}V(\rho',\phi')$$

Cylindrical geometry example:



$$\Phi(\rho = a, \phi, z) = V(\phi, z)$$

$$\Phi(\rho, \phi, z) = 0 \quad \text{on all other boundaries}$$

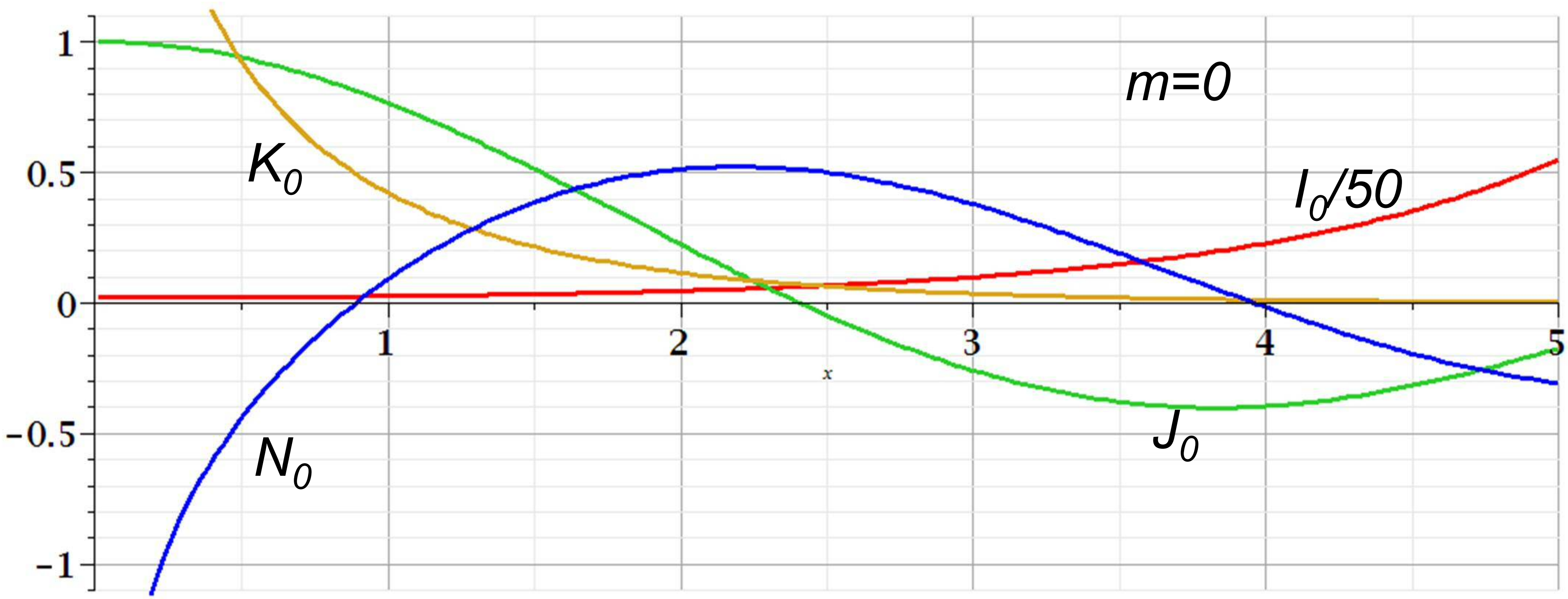
$$\Phi(\rho, \phi, z) = \sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin\left(m\phi + \alpha_{mn}\right)$$

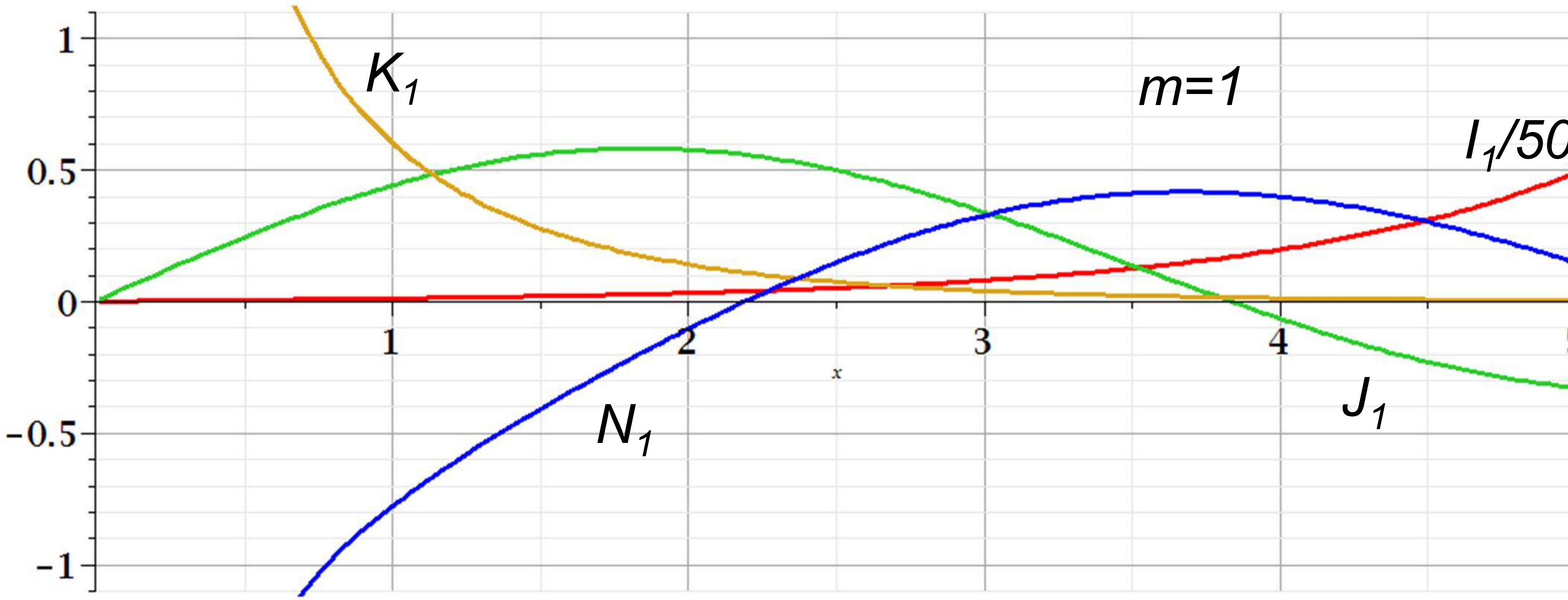
Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u), N_m(u), H_m(u)\equiv J_m(u)\pm N_m(u)$$

$$F_m^{-}(u)=I_m(u), K_m(u)$$





K_1
$m=1$
$I_1/50$
N_1
J_1
1
0.5
0
−0.5
−1
1
2
3
4
5
x

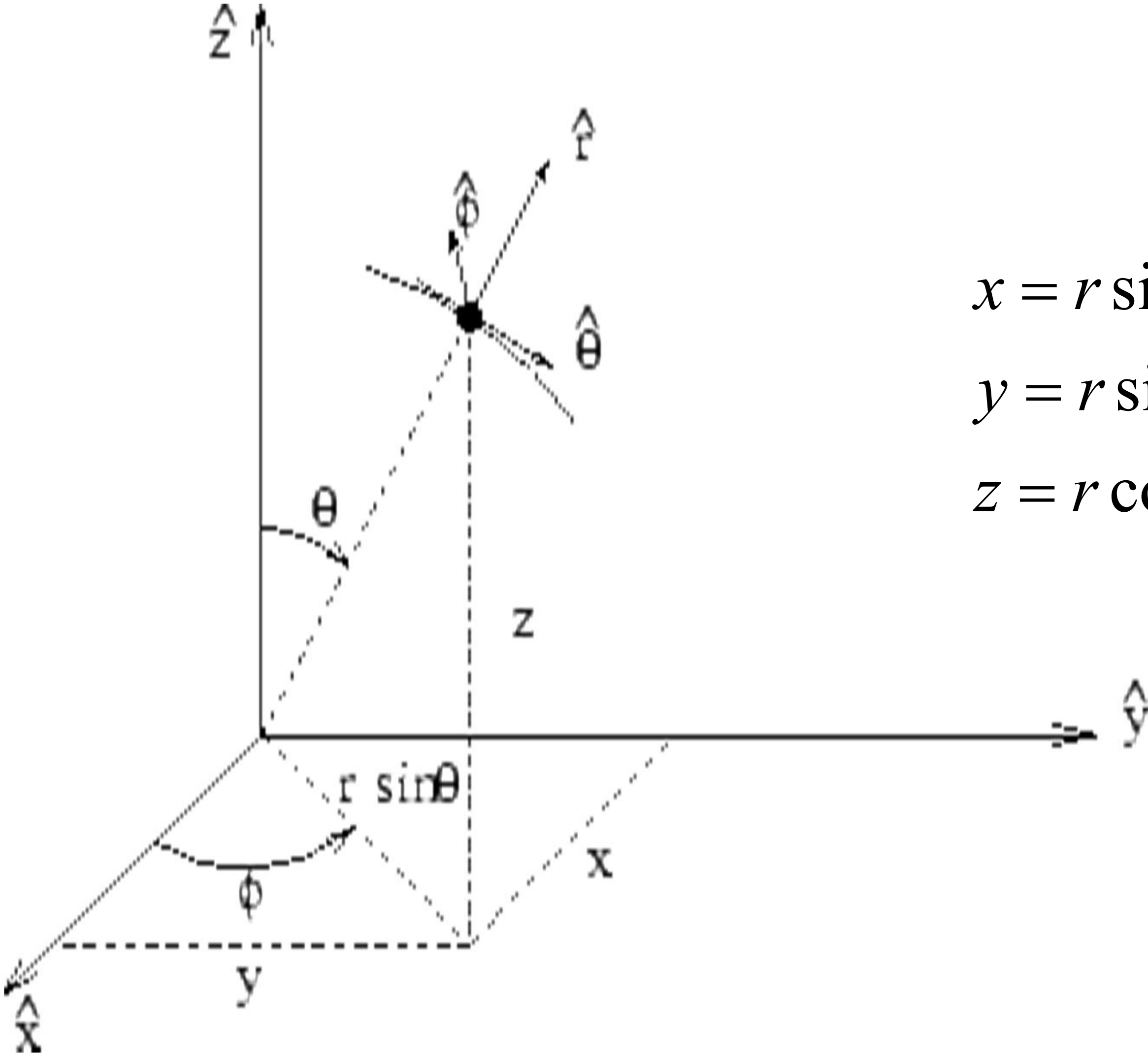
Some useful identities involving cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(1-\frac{m^2}{u^2}\right)\right)J_m(u)=0 \qquad \text{for integer } m$$

Properties of Bessel functions in terms of zeros: x_{mn} $\qquad J_m(x_{mn})=0$

$$\int_0^a \rho d\rho J_m(x_{mn}\rho/a)J_m(x_{mn'}\rho/a)=\frac{a^2}{2}\left(J_{m+1}(x_{mn})\right)^2\delta_{nn'}$$

Poisson and Laplace equation in spherical polar coordinates



$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

http://www.uic.edu/classes/eecs/eecs520/textbook/node32.html

Poisson and Laplace equation in spherical polar coordinates -- continued

Laplace equation for electrostatic potential $\Phi(r,\theta,\phi)$:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)\Phi=0$$

$$\Phi(r,\theta,\phi)=\sum_{lm}R_{lm}(r)Y_{lm}(\theta,\phi)$$

Spherical harmonic functions:

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi^2}\right)Y_{lm}(\theta,\phi)=-l(l+1)Y_{lm}(\theta,\phi)$$

Properties of spherical harmonic functions

$$Y_{lm}(\theta,\phi) = (-1)^m Y_{l(-m)}{}^*(\theta,\phi) \qquad \text{(standard Condon - Shortley convention)}$$

$$\int d\Omega Y_{lm}(\theta,\varphi) Y_{l'm'}{}^*(\theta,\varphi) \equiv \int \sin\theta \, d\theta \, d\varphi \; Y_{lm}(\theta,\varphi) Y_{l'm'}{}^*(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

Completeness:

$$\sum_{lm} Y_{lm}(\theta,\varphi) Y_{lm}{}^*(\theta',\varphi') = \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}') \equiv \delta(\cos\theta - \cos\theta')\delta(\phi - \phi')$$

Relationship to Legendre polynomials:

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^{\;*}(\theta',\varphi')$$

Example for isolated charge density $\rho(\mathbf{r})$ with electrostatic potential vanishing for $r \rightarrow \infty$:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0} \int d^3r' \rho(\mathbf{r}') \left(\sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^{\;*}(\theta',\varphi') \right)$$

Example -- continued

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\rho(\mathbf{r}')\left(\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_{lm}(\theta,\varphi)Y_{lm}^{\ *}(\theta',\varphi')\right)$$

$$\text{Suppose:}\quad \rho(\mathbf{r}') = \frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$$

$$\int d\Omega'\,Y_{lm}^{\ *}(\theta',\varphi') = \sqrt{4\pi}\,\delta_{l0}\delta_{m0}$$

$$\Phi(\mathbf{r}) = \frac{4\pi}{4\pi\varepsilon_0}\int_0^\infty r'^2\,dr'\,\frac{r_<^{\,0}}{r_>^{\,1}}\frac{Q}{a^3\pi^{3/2}}e^{-r'^2/a^2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{\mathrm{erf}(r/a)}{r}$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^{\ l}}{r_>^{\ l+1}}Y_{lm}(\theta,\varphi)Y_{lm}{}^*(\theta',\varphi')$$

Elements of "proof":

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r_>\left(1+\left(\frac{r_<}{r_>}\right)^2-2\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'\right)^{1/2}}=$$

$$=\frac{1}{r_>}\left(1+\left(\frac{r_<}{r_>}\right)\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'+\left(\frac{r_<}{r_>}\right)^2\left(\frac{3}{2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')^2-\frac{1}{2}\right)+\cdots\right)$$

$$=\frac{1}{r_>}\left(\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')\right)$$

Useful identity:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{lm} \frac{4\pi}{2l+1} \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^{\,*}(\theta',\varphi')$$

Elements of "proof" -- continued:

Sum rule for spherical harmonics:

$$P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\hat{\mathbf{r}})\, Y_{lm}^{\,*}(\hat{\mathbf{r}}')$$

Note that for $\hat{\mathbf{r}}=\hat{\mathbf{r}}'$, $P_l(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')=1$

$$\Rightarrow \frac{4\pi}{2l+1}\sum_{m=-l}^{l} \left|Y_{lm}(\hat{\mathbf{r}})\right|^2 = 1$$

Some spherical harmonic functions:

$$Y_{00}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1(\pm 1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2(\pm 2)}(\hat{\mathbf{r}}) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

$$Y_{2(\pm 1)}(\hat{\mathbf{r}}) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$$

$$Y_{20}(\hat{\mathbf{r}}) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$