PHY 712 Electrodynamics
11-11:50 AM MWF Olin 107

Plan for Lecture 6:

Continue reading Chapter 2

1. Methods of images -- planes, spheres
2. Solution of Poisson equation in for other geometries -- cylindrical

Course schedule for 2013

(Preliminary schedule -- subject to frequent adjustment.)

Date	JDJ Reading	Topic	Assign.
01-16(Wed)	Chap. 1	Introduction, units and Poisson equation.	#1
01-18(Fri)	Chap. 1	Electronstatic energy calculations	#2
01-21(Mon)	*No class*	*MKL Holiday*	
01-23(Wed)	Chap. 1	Poisson Equation and Green's Functions	#3
01-25(Fri)	Chap. 1 & 2	Green's Theorem and Functions	#4
01-28(Mon)	Chap. 1 & 2	Brief introduction to numerical methods	#5
01-30(Wed)	Chap. 2	Method of images	#6

TODAY --

WFU Joint Physics and Chemistry Colloquium

TITLE: Capturing pH mediated physics and chemistry in biological physics

SPEAKER: Professor Charles L. Brooks III,

Warner-Lambert/Parke-Davis Professor of Chemistry and Professor of Biophysics,
Department of Chemistry and Biophysics Program,
University of Michigan, Ann Arbor, Michigan

TIME: Wednesday January 30, 2013 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

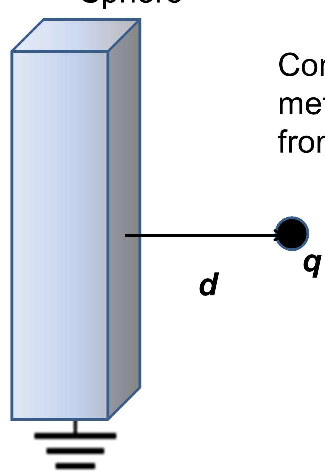
Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

Method of images

Clever trick for specialized geometries:

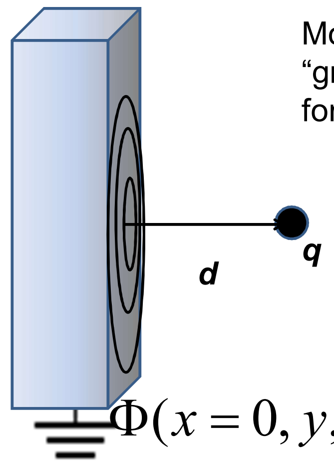
- Flat plane (surface)
- Sphere

Planar case:



Consider a grounded metal sheet, a distance ***d*** from a point charge ***q***.

A grounded metal sheet, a distance ***d*** from a point charge ***q***.



Mobile charges from the "ground" respond to the force from the charge ***q***.

$$\Phi(x = 0, y, z) = 0$$

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

$$\nabla^2\Phi = -\frac{q}{\varepsilon_0}\delta^3(\mathbf{r} - d\hat{\mathbf{x}})$$

$$\Phi(x = 0, y, z) = 0$$

Trick for $x \geq 0$:

$$\Phi(x \geq 0, y, z) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r} - d\hat{\mathbf{x}}|} - \frac{q}{|\mathbf{r} + d\hat{\mathbf{x}}|}\right)$$

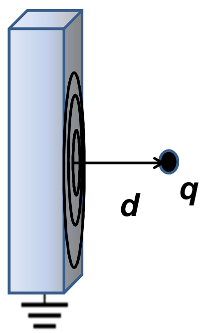
Surface charge density:

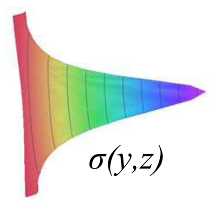
$$\sigma(y,z) = \varepsilon_0 E(0, y, z) = -\varepsilon_0 \frac{d\Phi(0, y, z)}{dx} = -\frac{q}{4\pi}\left(\frac{2d}{(d^2 + y^2 + z^2)^{3/2}}\right)$$

A grounded metal sheet, a distance ***d*** from a point charge ***q***.

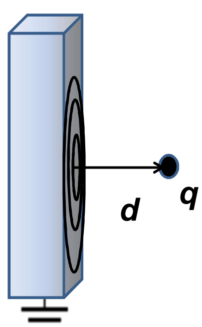
$$\text{Surface charge density:} \quad \sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

$$\text{Note:} \iint dydz \;\; \sigma(y,z) = -\frac{q2d}{4\pi}2\pi\int_0^\infty \frac{udu}{\left(d^2+u^2\right)^{3/2}} = -q$$





A grounded metal sheet, a distance ***d*** from a point charge ***q***.



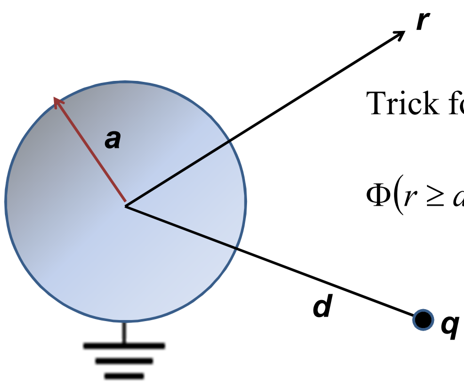
Surface charge density:

$$\sigma(y,z) = -\frac{q}{4\pi}\left(\frac{2d}{\left(d^2+y^2+z^2\right)^{3/2}}\right)$$

Force between charge and sheet:

$$\mathbf{F} = \frac{-q^2\hat{\mathbf{x}}}{4\pi\varepsilon_0(2d)^2}$$

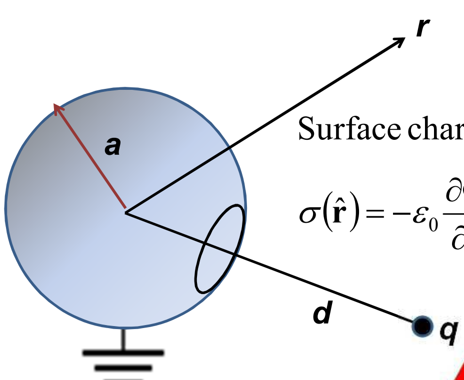
A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Trick for $r \geq a$:

$$\Phi(r \geq a) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{d}|} - \frac{q}{\frac{d}{a}\left|\mathbf{r}-\mathbf{d}\frac{a^2}{d^2}\right|}\right)$$

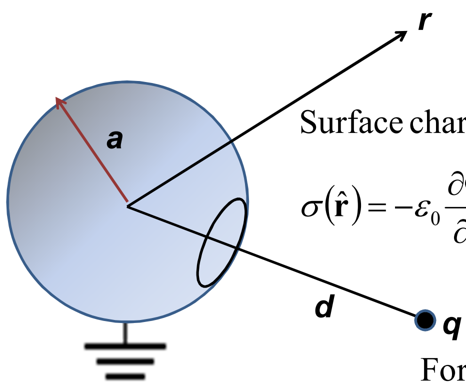
A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Surface charge density:

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \frac{\partial \Phi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

A grounded metal sphere of radius *a*, in the presence of a point charge ***q*** at a distance *d* from its center.



Surface charge density :

$$\sigma(\hat{\mathbf{r}}) = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = -\frac{q}{4\pi}\frac{a}{d}\frac{\left(1-\frac{a^2}{d^2}\right)}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}$$

Force between sphere and q :

$$|\mathbf{F}| = \frac{q^2}{4\pi\varepsilon_0}\frac{ad}{\left(d^2-a^2\right)^2}$$

Use of image charge formalism to construct Green's function

Example :

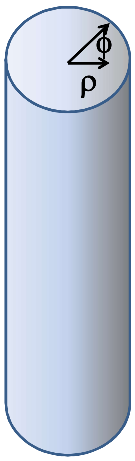
Suppose we have a Dirichlet boundary value problem on a sphere of radius a :

$$\nabla^2 \Phi = -\frac{\rho(\mathbf{r})}{\varepsilon_0} \qquad \Phi(r=a) = 0$$

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

$$\Rightarrow \text{For} \quad r, r' > a: \qquad G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{\frac{r'}{a}\left|\mathbf{r}-\frac{a^2}{r'^2}\mathbf{r}'\right|}$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



Corresponding orthogonal functions from solution of Laplace equation: $\nabla^2\Phi = 0$

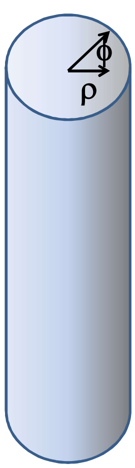
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

$$\Phi(\rho,\phi)=\Phi(\rho,\phi+m2\pi)$$

$\Rightarrow$ General solution of the Laplace equation in these coordinates:

$$\Phi(\rho,\phi)=\mathrm{A}_0+\mathrm{B}_0\ln(\rho)+\sum_{m=1}^{\infty}\left(A_m\rho^m+B_m\rho^{-m}\right)\sin(m\phi+\alpha_m)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with no *z*-dependence (infinitely long wire, for example):



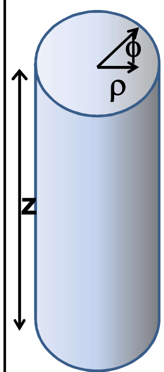
Green's function appropriate for this geometry with boundary conditions at $\rho=0$ and $\rho=\infty$:

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)G(\rho,\rho',\phi,\phi') =$$

$$-4\pi\frac{\delta(\rho-\rho')}{\rho}\delta(\cos\phi-\cos\phi')$$

$$G(\rho,\rho',\phi,\phi') = -\ln\left(\rho_>{}^2\right)+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos\left(m(\phi-\phi')\right)$$

Solution of the Poisson/Laplace equation in various geometries -- cylindrical geometry with *z*-dependence



Corresponding orthogonal functions from solution of Laplace equation: $\nabla^2\Phi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

$$\Phi(\rho,\phi,z)=\Phi(\rho,\phi+m2\pi,z)$$

$$\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$$

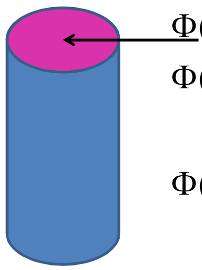
Cylindrical geometry continued:

$$\frac{d^2Z}{dz^2}-k^2Z=0 \qquad \Rightarrow Z(z)=\sinh(kz),\cosh(kz),e^{\pm kz}$$

$$\frac{d^2Q}{d\phi^2}+m^2Q=0 \qquad \Rightarrow Q(\phi)=e^{\pm im\phi}$$

$$\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}+\left(k^2-\frac{m^2}{\rho^2}\right)R=0 \qquad \Rightarrow J_m(k\rho),N_m(k\rho)$$

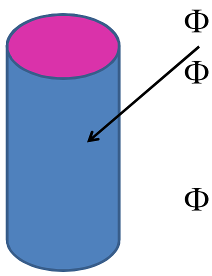
Cylindrical geometry example:



$$\Phi(\rho,\phi,z=L)=V(\rho,\phi)$$
$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} J_m(k_{mn}\rho)\sinh(k_{mn}z)\sin(m\phi+\alpha_{mn})$$

Cylindrical geometry example:



$$\Phi(\rho=a,\phi,z)=V(\phi,z)$$
$$\Phi(\rho,\phi,z)=0 \quad \text{on all other boundaries}$$

$$\Phi(\rho,\phi,z)=\sum_{n,m} A_{mn} I_m\left(\frac{n\pi\rho}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\sin(m\phi+\alpha_{mn})$$

Comments on cylindrical Bessel functions

$$\left(\frac{d^2}{du^2}+\frac{1}{u}\frac{d}{du}+\left(\pm 1-\frac{m^2}{u^2}\right)\right)F_m^{\pm}(u)=0$$

$$F_m^{+}(u)=J_m(u),N_m(u),H_m(u)\equiv J_m(u)\pm N_m(u)$$

$$F_m^{-}(u)=I_m(u),K_m(u)$$