PHY 712 Electrodynamics
11-11:50 AM MWF Olin 107

Plan for Lecture 27:

Continue reading Chap. 11 – Theory of Special Relativity

A. **Transformations of the Electromagnetic Fields**

B. **Connection to Liénard-Wiechert potentials for constant velocity sources**

	03-13(Wed)	*Spring Break*		
	03-15(Fri)	*Spring Break*		
	03-18(Mon)	*APS Meeting*	(no class)	Exam
	03-20(Wed)	*APS Meeting*	(no class)	Exam
	03-22(Fri)	*APS Meeting*	(no class)	Exam
25	03-25(Mon)	Chap. 11	Lorentz transformations	#17
26	03-27(Wed)	Chap. 11	Transformations between electromagnetic fields	#18
27	03-28(Thur)	Chap. 11	Liénard-Wiechert potentials revisited	
	03-29(Fri)	*Good Friday*	(no class)	
28	04-01(Mon)	Chap. 14	Radiation by accelerated charges	
29	04-03(Wed)	Chap. 14	Radiation by accelerated charges	
30	04-05(Fri)	Chap. 14	Synchrotron radiation spectrum	
31	04-08(Mon)			
32	04-10(Wed)			
33	04-12(Fri)			
	04-15(Mon)		(no class -- presentation preparation)	
	04-17(Wed)		(no class -- presentation preparation)	
	04-19(Fri)		(no class -- presentation preparation)	
34	04-22(Mon)			
35	04-24(Wed)			
36	04-26(Fri)			
	04-29(Mon)		Student presentations	
	05-01(Wed)		Student presentations	

Velocity transformation detail from Lecture 26:

$$\text{Consider}: \; u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \qquad u_y = \frac{u'_y}{\gamma_v\left(1 + vu'_x/c^2\right)} \qquad u_z = \frac{u'_z}{\gamma_v\left(1 + vu'_x/c^2\right)}.$$

$$\text{Note that} \qquad \gamma_u \equiv \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1 + vu'_x/c^2}{\sqrt{1-(u'/c)^2}\sqrt{1-(v/c)^2}}$$

Elements of "proof":

$$1 - \left(\frac{u}{c}\right)^2 = \frac{1}{\gamma_v^2\left(1+vu'_x/c^2\right)^2}\left(\gamma_v^2\left(1+vu'_x/c^2\right)^2 - \gamma_v^2\left(u'_x/c + v/c\right)^2 - \left(u'_y/c\right)^2 - \left(u'_z/c\right)^2\right)$$

$$= \frac{1}{\gamma_v^2\left(1+vu'_x/c^2\right)^2}\left(\gamma_v^2\left(1-(v/c)^2\right)\left(1-(u'_x/c)^2\right) - \left(u'_y/c\right)^2 - \left(u'_z/c\right)^2\right)$$

$$= \frac{1-(u'/c)^2}{\gamma_v^2\left(1+vu'_x/c^2\right)^2}$$

Special theory of relativity and Maxwell's equations

Continuity equation : $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Lorentz gauge condition : $$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

Potential equations : $$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} + \nabla^2 \Phi = 4\pi\rho$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

Field relations : $$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

More 4-vectors:

$$\text{Time and position}: \quad \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Rightarrow x^{\alpha}$$

$$\text{Charge and current}: \quad \begin{pmatrix} c\rho \\ J_x \\ J_y \\ J_z \end{pmatrix} \Rightarrow J^{\alpha}$$

$$\text{Vector and scalar potentials}: \quad \begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow A^{\alpha}$$

Lorentz transformations

$$\mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{Time and space:} \qquad x^\alpha = \mathcal{L}_v x'^\alpha \equiv \mathcal{L}_v{}^{\alpha\beta} x'^\beta$$

$$\text{Charge and current:} \qquad J^\alpha = \mathcal{L}_v J'^\alpha \equiv \mathcal{L}_v{}^{\alpha\beta} J'^\beta$$

$$\text{Vector and scalar potential:} \qquad A^\alpha = \mathcal{L}_v A'^\alpha \equiv \mathcal{L}_v{}^{\alpha\beta} A'^\beta$$

4-vector relationships

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \Leftrightarrow \left(A^0, \mathbf{A}\right): \text{ upper index 4 - vector } \quad A^\alpha \text{ for } \left(\alpha = 0,1,2,3\right)$$

Keeping track of signs -- lower index 4 - vector $A_\alpha = \left(A^0, -\mathbf{A}\right)$

Derivative operators:

$$\partial^\alpha = \left(\frac{\partial}{c\partial t}, -\nabla\right) \qquad \partial_\alpha = \left(\frac{\partial}{c\partial t}, \nabla\right)$$

Special theory of relativity and Maxwell's equations

Continuity equation: $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$ $\qquad \partial_\alpha J^\alpha = 0$

Lorentz gauge condition: $\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$ $\qquad \partial_\alpha A^\alpha = 0$

Potential equations:

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi\rho$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

$$\partial_\alpha \partial^\alpha A^\alpha = \frac{4\pi}{c} J^\alpha$$

Field relations:

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Electric and Magnetic field relationships

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$E_x = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{c\partial t} = -\left(\partial^0 A^1 - \partial^1 A^0\right)$$

$$E_y = -\frac{\partial \Phi}{\partial y} - \frac{\partial A_y}{c\partial t} = -\left(\partial^0 A^2 - \partial^2 A^0\right)$$

$$E_z = -\frac{\partial \Phi}{\partial z} - \frac{\partial A_z}{c\partial t} = -\left(\partial^0 A^3 - \partial^3 A^0\right)$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = -\left(\partial^2 A^3 - \partial^3 A^2\right)$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = -\left(\partial^3 A^1 - \partial^1 A^3\right)$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -\left(\partial^1 A^2 - \partial^2 A^1\right)$$

Field strength tensor $F^{\alpha\beta} \equiv \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)$

$$F^{\alpha\beta} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Transformation of field strength tensor

$$F^{\alpha\beta} = \mathcal{L}_v{}^{\alpha\gamma} F'^{\gamma\delta} \mathcal{L}_v{}^{\delta\beta} \qquad \mathcal{L}_v = \begin{pmatrix} \gamma_v & \gamma_v\beta_v & 0 & 0 \\ \gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y + \beta_v B'_z\right) & -\gamma_v\left(E'_z - \beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z + \beta_v E'_y\right) & \gamma_v\left(B'_y - \beta_v E'_z\right) \\ \gamma_v\left(E'_y + \beta_v B'_z\right) & \gamma_v\left(B'_z + \beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z - \beta_v B'_y\right) & -\gamma_v\left(B'_y - \beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

Inverse transformation of field strength tensor

$$F'^{\alpha\beta} = \mathcal{L}_v^{-1\alpha\gamma} F^{\gamma\delta} \mathcal{L}_v^{-1\delta\beta} \qquad \mathcal{L}_v^{-1} = \begin{pmatrix} \gamma_v & -\gamma_v\beta_v & 0 & 0 \\ -\gamma_v\beta_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & -E'_x & -\gamma_v\left(E'_y - \beta_v B'_z\right) & -\gamma_v\left(E'_z + \beta_v B'_y\right) \\ E'_x & 0 & -\gamma_v\left(B'_z - \beta_v E'_y\right) & \gamma_v\left(B'_y + \beta_v E'_z\right) \\ \gamma_v\left(E'_y - \beta_v B'_z\right) & \gamma_v\left(B'_z - \beta_v E'_y\right) & 0 & -B'_x \\ \gamma_v\left(E'_z + \beta_v B'_y\right) & -\gamma_v\left(B'_y + \beta_v E'_z\right) & B'_x & 0 \end{pmatrix}$$

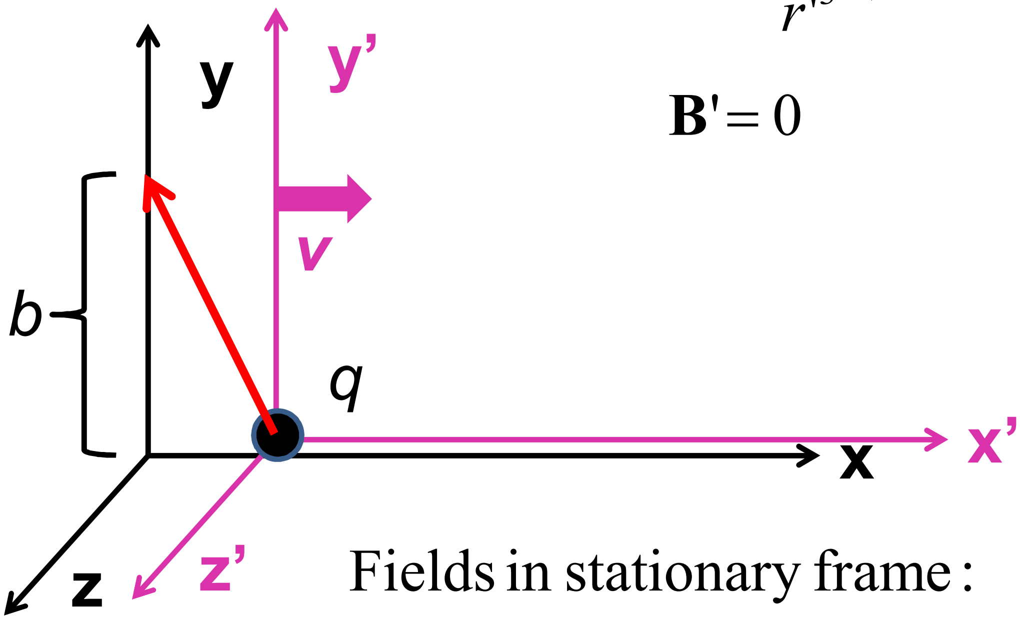
Summary of results:

$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$

$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

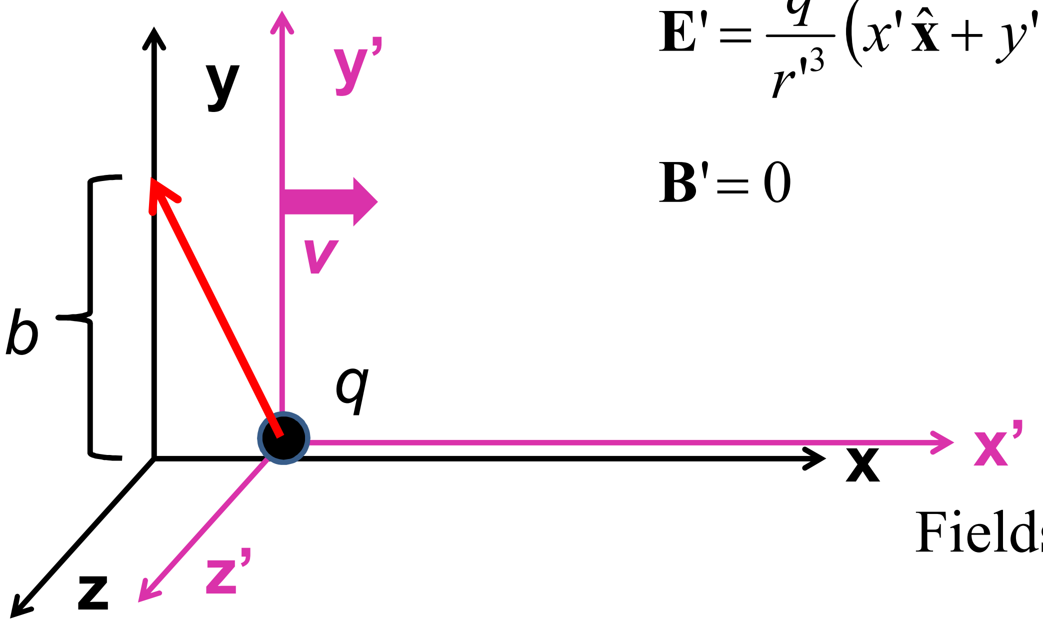
Fields in stationary frame:

$$E_x = E'_x \qquad B_x = B'_x$$

$$E_y = \gamma_v\left(E'_y + \beta_v B'_z\right) \qquad B_y = \gamma_v\left(B'_y - \beta_v E'_z\right)$$

$$E_z = \gamma_v\left(E'_z - \beta_v B'_y\right) \qquad B_z = \gamma_v\left(B'_z + \beta_v E'_y\right)$$

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

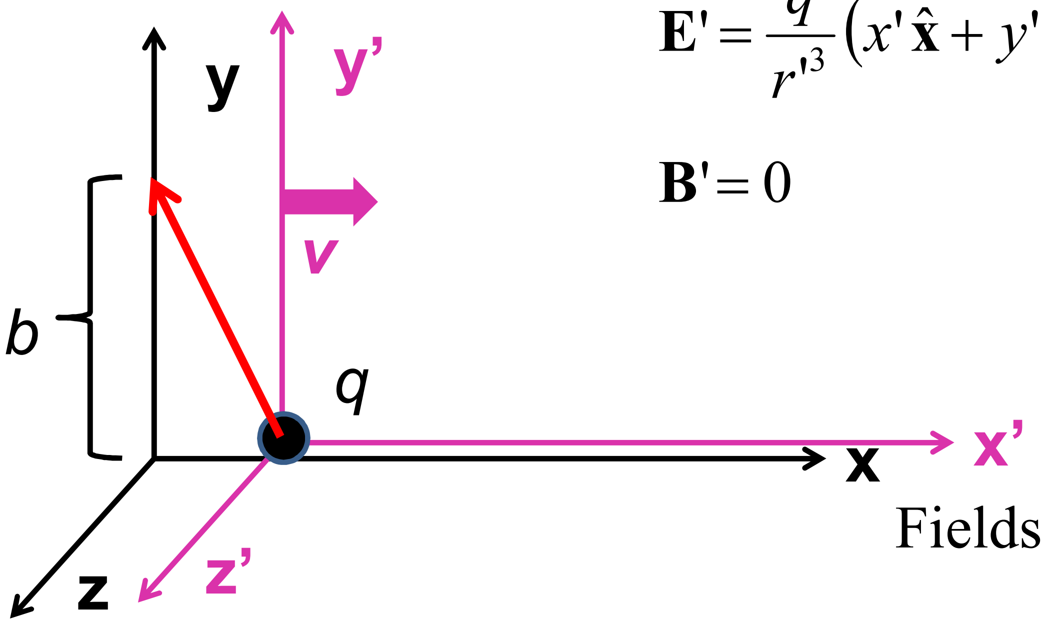
Fields in stationary frame:

$$E_x = E'_x = \frac{q\left(-vt'\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

Example:



Fields in moving frame:

$$\mathbf{E}' = \frac{q}{r'^3}\left(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}}\right) = \frac{q\left(-vt'\hat{\mathbf{x}} + b\hat{\mathbf{y}}\right)}{\left(\left(-vt'\right)^2 + b^2\right)^{3/2}}$$

$$\mathbf{B}' = 0$$

Fields in stationary frame:

$$E_x = E'_x = \frac{q\left(-v\gamma_v t\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$E_y = \gamma_v\left(E'_y\right) = \frac{q\left(\gamma_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

$$B_z = \gamma_v\left(\beta_v E'_y\right) = \frac{q\left(\gamma_v \beta_v b\right)}{\left(\left(-v\gamma_v t\right)^2 + b^2\right)^{3/2}}$$

Expression in terms of consistent coordinates

$$E_y = \frac{q(\gamma_v b)}{\left((-v\gamma_v t)^2 + b^2\right)^{3/2}}$$