# PHY 712 Electrodynamics
# 11-11:50 AM MWF Olin 107

## Plan for Lecture 24:

**Continue reading Chap. 9**

A. **Electromagnetic waves due to specific sources**

B. **Dipole radiation examples**

| | | | | |
|---|---|---|---|---|
| **13** | 02-15(Fri) | Chap. 5 | Magnetic dipole fields | Exam |
| **14** | 02-18(Mon) | Chap. 5 | Permeable media | Exam |
| **15** | 02-20(Wed) | Chap. 5 | Magnetic susceptibility and permeability | Exam |
| **16** | 02-22(Fri) | Chap. 6 | Maxwell's equations | Exam |
| **17** | 02-25(Mon) | Chap. 6 | Poynting Vector | #11 |
| **18** | 02-27(Wed) | Chap. 7 | Reflectance and transmittance of electromagnetic plane waves | #12 |
| **19** | 02-28(Thur) | Chap. 7 | Anisotropic media | #13 |
| **20** | 03-01(Fri) | Chap. 7 | Dielectric models; Kramers-Kronig Relations | #14 |
| **21** | 03-04(Mon) | Chap. 8 | TEM modes | #15 |
| **22** | 03-06(Wed) | Chap. 8 | TE and TM modes | #16 |
| **23** | 03-07(Thur) | Chap. 9 | Sources of electromagnetic waves | |
| **4** | 03-08(Fri) | Chap. 9 | Dipole radiation patterns | |
| | 03-11(Mon) | *Spring Break* | | |
| | 03-13(Wed) | *Spring Break* | | |
| | 03-15(Fri) | *Spring Break* | | |
| | 03-18(Mon) | *APS Meeting* | (no class) | Exam |
| | 03-20(Wed) | *APS Meeting* | (no class) | Exam |
| | 03-22(Fri) | *APS Meeting* | (no class) | Exam |
| **25** | 03-25(Mon) | Chap. 9 | | |
| **26** | 03-27(Wed) | Chap. 9 | | |
| **27** | 03-28(Thur) | | | |
| | 03-29(Fri) | *Good Friday* | | |
| **28** | 04-01(Mon) | | | |

**Reminders**:

- Topic choice for "computational project" due Friday 3/8/2013 (via email or written on paper)
- Outstanding homework due before 3/13/2013 (when mid term grades are due)
- Exam 2 will be available after 3/13/2013 and is due 3/25/3013

# Maxwell's equations

Microscopic or vacuum form $(\mathbf{P}=0;\ \mathbf{M}=0)$:

Coulomb's law: $\nabla\cdot\mathbf{E}=\rho/\varepsilon_0$

Ampere-Maxwell's law: $\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$

Faraday's law: $\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$

No magnetic monopoles: $\nabla\cdot\mathbf{B}=0$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials:

$$\text{Lorentz gauge form -- require:} \quad \nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Note that the Lorentz gauge is consistent with the source continuity condition: $\dfrac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r},t) = 0$

## Electromagnetic waves from time harmonic sources

Charge density: $\rho(\mathbf{r},t)=\Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Current density: $\mathbf{J}(\mathbf{r},t)=\Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

$\Rightarrow$ Scalar potential: $\Phi(\mathbf{r},t)=\Re\left(\tilde{\Phi}(\mathbf{r},\omega)e^{-i\omega t}\right)$

$\Rightarrow$ Vector potential: $\mathbf{A}(\mathbf{r},t)=\Re\left(\tilde{\mathbf{A}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

For $k\equiv\frac{\omega}{c}$:

$$\tilde{\Phi}(\mathbf{r},\omega)=\tilde{\Phi}_0(\mathbf{r},\omega)+\frac{1}{4\pi\varepsilon_0}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\rho}(\mathbf{r}',\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\tilde{\mathbf{A}}_0(\mathbf{r},\omega)+\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Electromagnetic waves from time harmonic sources – continued:

Useful expansion:

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \tilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm}\tilde{\phi}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0}\int d^3r'\,\tilde{\rho}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = \tilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm}\tilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\tilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0\int d^3r'\,\tilde{\mathbf{J}}(\mathbf{r}',\omega)\,j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}}')$$

Forms of spherical Bessel and Hankel functions:

$$j_0(x) = \frac{\sin(x)}{x} \qquad h_0(x) = \frac{e^{ix}}{ix}$$

$$j_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x} \qquad h_1(x) = -\left(1 + \frac{i}{x}\right)\frac{e^{ix}}{x}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin(x) - \frac{3\cos(x)}{x^2} \qquad h_2(x) = i\left(1 + \frac{3i}{x} - \frac{3}{x^2}\right)\frac{e^{ix}}{x}$$

Assymptotic behavior:

$$x << 1 \qquad \Rightarrow j_l(x) \approx \frac{(x)^l}{(2l+1)!!}$$

$$x >> 1 \qquad \Rightarrow h_l(x) \approx (-i)^{l+1}\frac{e^{ix}}{x}$$

Electromagnetic waves from time harmonic sources – continued:

Dipole radiation case:

Define dipole moment at frequency $\omega$:

$$\mathbf{p}(\omega) \equiv \int d^3r\, \mathbf{r}\tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r\, \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

For fields outside extent of source and $kr' << 1$ within the source:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i}{4\pi\omega\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

Electromagnetic waves from time harmonic sources – continued:

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla\widetilde{\Phi}(\mathbf{r},\omega) + i\omega\widetilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2((\hat{\mathbf{r}}\times\mathbf{p}(\omega))\times\hat{\mathbf{r}}) + \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2}\right)(1-ikr)\right)$$

$$\widetilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla\times\widetilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2(\hat{\mathbf{r}}\times\mathbf{p}(\omega))\left(1-\frac{1}{ikr}\right)$$

Power radiated for $kr >> 1$:

$$\frac{dP}{d\Omega} = r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg} = \frac{r^2}{2\mu_0}\hat{\mathbf{r}}\cdot\Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)\times\widetilde{\mathbf{B}}^*(\mathbf{r},\omega)\right)$$

$$= \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|(\hat{\mathbf{r}}\times\mathbf{p}(\omega))\times\hat{\mathbf{r}}\right|^2$$

Properties of dipole radiation field for *kr* >>1:

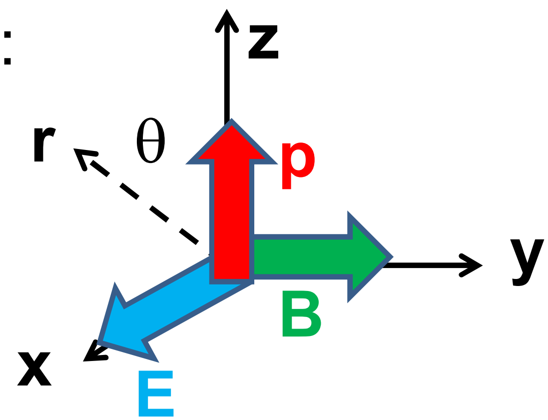
$$\tilde{\mathbf{E}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left(k^2\left(\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right)\right)$$

$$\tilde{\mathbf{B}}(\mathbf{r},\omega)=\frac{1}{4\pi\varepsilon_0 c^2}\frac{e^{ikr}}{r}k^2\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)$$

Power radiated for $kr>>1$:

$$\frac{dP}{d\Omega}=r^2\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle_{avg}=\frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\varepsilon_0}}\left|\left(\hat{\mathbf{r}}\times\mathbf{p}(\omega)\right)\times\hat{\mathbf{r}}\right|^2$$

Example:



## Example of dipole radiation source

$$\tilde{\mathbf{J}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0 e^{-r/R} \qquad \tilde{\rho}(\mathbf{r},\omega)=\frac{J_0}{-i\omega R}\cos\theta e^{-r/R}$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\,(ik\mu_0)\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_0(kr_>)j_0(kr_<)$$

$$\tilde{\Phi}(\mathbf{r},\omega)=-\frac{J_0 k}{\varepsilon_0\omega R}\cos\theta\int_0^\infty r'^2\,dr'\,e^{-r'/R}h_1(kr_>)j_1(kr_<)$$

Evaluation for $r >> R$:

$$\tilde{\mathbf{A}}(\mathbf{r},\omega)=\hat{\mathbf{z}}J_0\mu_0\frac{e^{ikr}}{r}\;\frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\tilde{\Phi}(\mathbf{r},\omega)=\frac{J_0 k}{\varepsilon_0\omega}\cos\theta\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\frac{2R^3}{\left(1+k^2R^2\right)^2}$$

## Example of dipole radiation source -- continued

Evaluation for $r >> R$:

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \hat{\mathbf{z}} J_0 \mu_0 \frac{e^{ikr}}{r} \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

$$\widetilde{\Phi}(\mathbf{r},\omega) = \frac{J_0 k}{\varepsilon_0 \omega} \cos\theta \frac{e^{ikr}}{r} \left(1 + \frac{i}{kr}\right) \frac{2R^3}{\left(1+k^2R^2\right)^2}$$

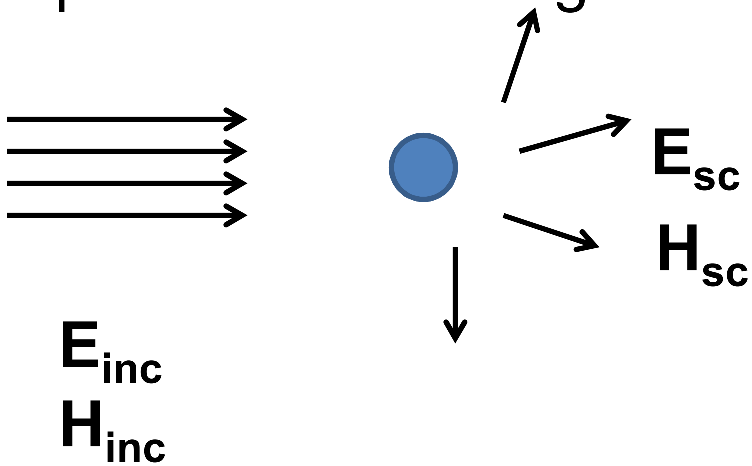
Relationship to pure dipole approximation (exact when $kR \rightarrow 0$)

$$\mathbf{p}(\omega) \equiv \int d^3r\, \mathbf{r}\widetilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega}\int d^3r\, \widetilde{\mathbf{J}}(\mathbf{r},\omega) = -\frac{8\pi R^3 J_0}{i\omega}\hat{\mathbf{z}}$$

Corresponding dipole fields: $\widetilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi}\mathbf{p}(\omega)\frac{e^{ikr}}{r}$

$$\widetilde{\Phi}(\mathbf{r},\omega) = -\frac{i}{4\pi\omega\varepsilon_0}\mathbf{p}(\omega)\cdot\hat{\mathbf{r}}\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{r}$$

# Dipole radiation in light scattering by small (dielectric) particles

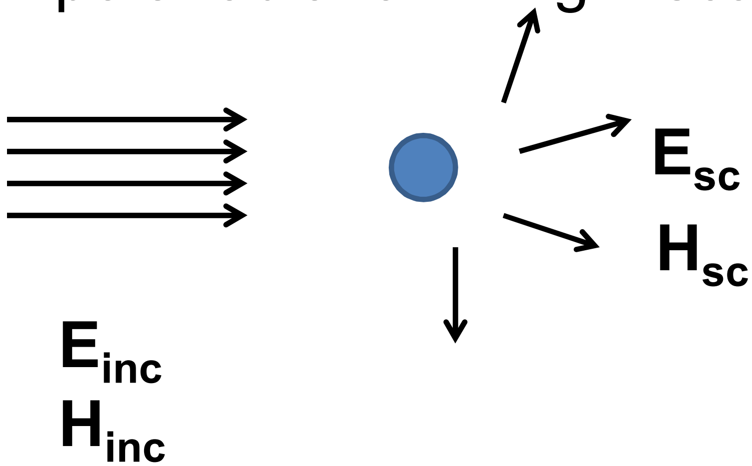



$$\mathbf{E}_{\text{inc}} = \hat{\boldsymbol{\varepsilon}}_0 E_0 e^{ik\hat{\mathbf{k}}_0\cdot\mathbf{r}} \qquad \mathbf{H}_{\text{inc}} = \frac{1}{\mu_0 c}\hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$$

In electric dipole approximation :

$$\mathbf{E}_{\text{sc}} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left(\left(\hat{\mathbf{r}}\times\mathbf{p}\right)\times\hat{\mathbf{r}}\right) \qquad \mathbf{H}_{\text{sc}} = \frac{1}{\mu_0 c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{sc}}$$

Dipole radiation in light scattering by small (dielectric) particles




Scattering cross section :

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_{\mathbf{0}},\hat{\boldsymbol{\varepsilon}}_0\right)=\frac{r^2\hat{\mathbf{r}}\cdot\left\langle\mathbf{S}_{sc}\right\rangle_{avg}}{\hat{\mathbf{k}}_{\mathbf{0}}\cdot\left\langle\mathbf{S}_{inc}\right\rangle_{avg}}$$

$$=\frac{r^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0\cdot\mathbf{E}_{inc}\right|^2}=\frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2}\left|\hat{\boldsymbol{\varepsilon}}\cdot\mathbf{p}\right|^2$$

Estimation of scattering dipole moment:

Suppose the scattering particle is a dielectric sphere with permittivity $\varepsilon$ and radius *a*:

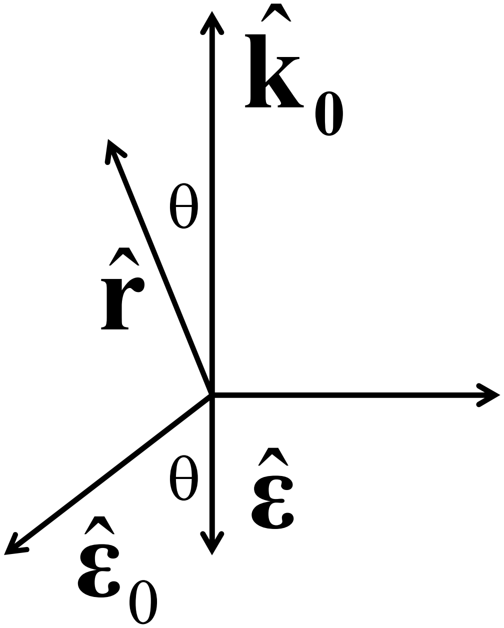
$$\mathbf{p} = 4\pi a^3 \varepsilon_0 \left( \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right) \mathbf{E}_{inc}$$

Scattering cross section:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\varepsilon}}; \hat{\mathbf{k}}_{\mathbf{0}}, \hat{\boldsymbol{\varepsilon}}_0\right) = \frac{r^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_{sc}\right|^2}{\left|\hat{\boldsymbol{\varepsilon}}_0 \cdot \mathbf{E}_{inc}\right|^2} = \frac{k^4}{\left(4\pi\varepsilon_0 E_0\right)^2} \left|\hat{\boldsymbol{\varepsilon}} \cdot \mathbf{p}\right|^2$$

$$= k^4 a^6 \left| \frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right|^2 \left|\hat{\boldsymbol{\varepsilon}} \cdot \hat{\boldsymbol{\varepsilon}}_0\right|^2$$

Scattering by dielectric sphere with permittivity ε and radius *a*:

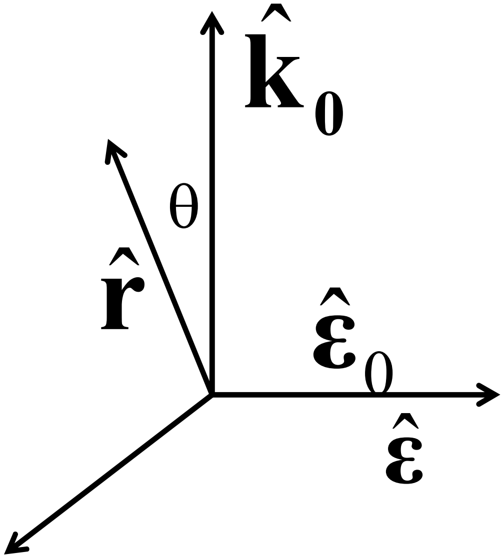



For $\mathbf{E}_{inc}$ polarized in scattering plane:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right)== k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \left|\hat{\boldsymbol{\varepsilon}}\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$= k^4 a^6 \left|\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right|^2 \cos^2\theta$$

Scattering by dielectric sphere with permittivity ε and radius *a*:




For $\mathbf{E}_{inc}$ polarized perpendicular to scattering plane:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right)=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left|\hat{\boldsymbol{\varepsilon}}\cdot\hat{\boldsymbol{\varepsilon}}_0\right|^2$$

$$=k^4a^6\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2$$

Assuming both polarizations are equally likely, average cross section is given by:

$$\frac{d\sigma}{d\Omega}\left(\hat{\mathbf{r}},\hat{\boldsymbol{\varepsilon}};\hat{\mathbf{k}}_0,\hat{\boldsymbol{\varepsilon}}_0\right)=\frac{k^4a^6}{2}\left|\frac{\varepsilon/\varepsilon_0-1}{\varepsilon/\varepsilon_0+2}\right|^2\left(\cos^2\theta+1\right)$$