PHY 712 Electrodynamics
11-11:50 AM MWF Olin 107

Plan for Lecture 23:

Start reading Chap. 9

A. Electromagnetic waves due to specific sources

B. Dipole radiation patterns

13	02-15(Fri)	Chap. 5	Magnetic dipole fields	Exam
14	02-18(Mon)	Chap. 5	Permeable media	Exam
15	02-20(Wed)	Chap. 5	Magnetic susceptibility and permeability	Exam
16	02-22(Fri)	Chap. 6	Maxwell's equations	Exam
17	02-25(Mon)	Chap. 6	Poynting Vector	#11
18	02-27(Wed)	Chap. 7	Reflectance and transmittance of electromagnetic plane waves	#12
19	02-28(Thur)	Chap. 7	Anisotropic media	#13
20	03-01(Fri)	Chap. 7	Dielectric models; Kramers-Kronig Relations	#14
21	03-04(Mon)	Chap. 8	TEM modes	#15
22	03-06(Wed)	Chap. 8	TE and TM modes	#16
23	03-07(Thur)	Chap. 9	Sources of electromagnetic waves	
24	03-08(Fri)	Chap. 9	Dipole radiation patterns	
	03-11(Mon)	Spring Break		
	03-13(Wed)	Spring Break		
	03-15(Fri)	Spring Break		
	03-18(Mon)	APS Meeting	(no class)	Exam
	03-20(Wed)	APS Meeting	(no class)	Exam
	03-22(Fri)	APS Meeting	(no class)	Exam
25	03-25(Mon)	Chap. 9		
26	03-27(Wed)	Chap. 9		
27	03-28(Thur)			
	03-29(Fri)	Good Friday		
28	04-01(Mon)			

Reminders:
- ➢ Topic choice for "computational project" due Friday 3/8/2013 (via email or written on paper)
- ➢ Outstanding homework due before 3/13/2013 (when mid term grades are due)
- ➢ Exam 2 will be available after 3/13/2013 and is due 3/25/3013

Maxwell's equations

Microscopic or vacuum form $(\mathbf{P} = 0; \; \mathbf{M} = 0)$:

Coulomb's law : $\qquad\qquad\qquad \nabla \cdot \mathbf{E} = \rho / \varepsilon_0$

Ampere - Maxwell's law : $\quad \nabla \times \mathbf{B} - \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$

Faraday's law : $\qquad\qquad\quad \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles : $\quad \nabla \cdot \mathbf{B} = 0$

$$\Rightarrow c^2 = \dfrac{1}{\varepsilon_0 \mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \boxed{\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

$$\text{Lorentz gauge form -- require :} \quad \nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \Phi_L}{\partial t} = 0$$

$$-\nabla^2 \Phi_L + \frac{1}{c^2} \frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

$$\text{General equation form :}$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Psi = -4\pi f$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - (t - |\mathbf{r}-\mathbf{r}'|/c)\right)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)f(\mathbf{r}',t')$$

Electromagnetic waves from time harmonic sources

Charge density : $\rho(\mathbf{r},t) = \Re\left(\tilde{\rho}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Current density : $\mathbf{J}(\mathbf{r},t) = \Re\left(\tilde{\mathbf{J}}(\mathbf{r},\omega)e^{-i\omega t}\right)$

Note that the continuity condition :

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r},t) = 0 \Rightarrow -i\omega\tilde{\rho}(\mathbf{r},\omega) + \nabla \cdot \tilde{\mathbf{J}}(\mathbf{r},\omega) = 0$$

General source : $f(\mathbf{r},t) = \Re\left(\tilde{f}(\mathbf{r},\omega)e^{-i\omega t}\right)$

For $$\tilde{f}(\mathbf{r},\omega) = \frac{1}{4\pi\varepsilon_0}\tilde{\rho}(\mathbf{r},\omega)$$

or $$\tilde{f}(\mathbf{r},\omega) = \frac{\mu_0}{4\pi}\tilde{J}_i(\mathbf{r},\omega)$$

Electromagnetic waves from time harmonic sources –
continued:

$$\Psi(\mathbf{r},t)=\Psi_{f=0}(\mathbf{r},t)+$$

$$\int d^3r'\int dt'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)f(\mathbf{r}',t')$$

$$\widetilde{\Psi}(\mathbf{r},\omega)e^{-i\omega t}=\widetilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t}+$$

$$\int d^3r'\int dt'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)\widetilde{f}(\mathbf{r}',\omega)e^{-i\omega t'}$$

$$=\widetilde{\Psi}_{f=0}(\mathbf{r},\omega)e^{-i\omega t}+\int d^3r'\frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\widetilde{f}(\mathbf{r}',\omega)e^{-i\omega t}$$

Electromagnetic waves from time harmonic sources – continued:

For scalar potential (Lorentz gauge, $k \equiv \dfrac{\omega}{c}$)

$$\widetilde{\Phi}(\mathbf{r},\omega) = \widetilde{\Phi}_0(\mathbf{r},\omega) + \frac{1}{4\pi\varepsilon_0} \int d^3 r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \widetilde{\rho}(\mathbf{r}',\omega)$$

For vector potential (Lorentz gauge, $k \equiv \dfrac{\omega}{c}$)

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \widetilde{\mathbf{A}}_0(\mathbf{r},\omega) + \frac{\mu_0}{4\pi} \int d^3 r' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \widetilde{\mathbf{J}}(\mathbf{r}',\omega)$$

Electromagnetic waves from time harmonic sources –
continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = ik \sum_{lm} j_l(kr_<) h_l(kr_>) Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}')$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + i n_l(kr)$

$$\widetilde{\Phi}(\mathbf{r},\omega) = \widetilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \widetilde{\phi}_{lm}(r,\omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0} \int d^3r' \, \widetilde{\rho}(\mathbf{r}',\omega) j_l(kr_<) h_l(kr_>) Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources –
continued:

Useful expansion :

$$\frac{e^{ik|\mathbf{r}-\mathbf{r'}|}}{4\pi|\mathbf{r}-\mathbf{r'}|} = ik\sum_{lm} j_l(kr_<)h_l(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}'})$$

Spherical Bessel function : $j_l(kr)$

Spherical Hankel function : $h_l(kr) = j_l(kr) + in_l(kr)$

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \widetilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm} \widetilde{\mathbf{a}}_{lm}(r,\omega)Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0 \int d^3r' \widetilde{\mathbf{J}}(\mathbf{r}',\omega)j_l(kr_<)h_l(kr_>)Y^*_{lm}(\hat{\mathbf{r}'})$$

Electromagnetic waves from time harmonic sources – continued:

$$\widetilde{\Phi}(\mathbf{r},\omega) = \widetilde{\Phi}_0(\mathbf{r},\omega) + \sum_{lm} \widetilde{\phi}_{lm}(r,\omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\phi}_{lm}(r,\omega) = \frac{ik}{\varepsilon_0} \int d^3r' \, \widetilde{\rho}(\mathbf{r}',\omega) j_l(kr_<) h_l(kr_>) Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\widetilde{\mathbf{A}}(\mathbf{r},\omega) = \widetilde{\mathbf{A}}_0(\mathbf{r},\omega) + \sum_{lm} \widetilde{\mathbf{a}}_{lm}(r,\omega) Y_{lm}(\hat{\mathbf{r}})$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) = ik\mu_0 \int d^3r' \, \widetilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr_<) h_l(kr_>) Y^*_{lm}(\hat{\mathbf{r}}')$$

For $r \gg$ (extent of source)

$$\widetilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \widetilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) \approx ik\mu_0 h_l(kr) \int d^3r' \, \widetilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr') Y^*_{lm}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources –
continued:

For $r \gg$ (extent of source)

$$\widetilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \widetilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$\widetilde{\mathbf{a}}_{lm}(r,\omega) \approx ik\mu_0 h_l(kr) \int d^3r' \, \widetilde{\mathbf{J}}(\mathbf{r}',\omega) j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}')$$

Note that $\widetilde{\rho}(\mathbf{r}',\omega)$ and $\widetilde{\mathbf{J}}(\mathbf{r}',\omega)$ are connected via the

continuity condition : $\quad -i\omega\,\widetilde{\rho}(\mathbf{r},\omega) + \nabla \cdot \widetilde{\mathbf{J}}(\mathbf{r},\omega) = 0$

$$\widetilde{\phi}_{lm}(r,\omega) \approx \frac{ik}{\varepsilon_0} h_l(kr) \int d^3r' \, \widetilde{\rho}(\mathbf{r}',\omega) j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}')$$

$$= -\frac{k}{\omega\varepsilon_0} h_l(kr) \int d^3r' \, \widetilde{\mathbf{J}}(\mathbf{r}',\omega) \cdot \nabla' \left(j_l(kr') Y^*{}_{lm}(\hat{\mathbf{r}}') \right)$$

Electromagnetic waves from time harmonic sources – continued:

Various approximations :

$$kr >> 1 \quad \Rightarrow h_l(kr) \approx (-i)^{l+1} \frac{e^{ikr}}{kr}$$

$$kr' << 1 \quad \Rightarrow j_l(kr') \approx \frac{(kr')^l}{(2l+1)!!}$$

Lowest (non - trivial) contributions in l expansions :

$$\widetilde{\phi}_{1m}(r,\omega) \approx -\frac{ik}{\varepsilon_0} \frac{e^{ikr}}{kr} \int d^3r' \, \widetilde{\rho}(\mathbf{r}',\omega) \frac{kr'}{3} Y^*{}_{1m}(\hat{\mathbf{r}}')$$

$$\widetilde{\mathbf{a}}_{00}(r,\omega) \approx ik\mu_0 \left((-i)\frac{e^{ikr}}{kr} \right) \int d^3r' \, \widetilde{\mathbf{J}}(\mathbf{r}',\omega) Y^*{}_{00}(\hat{\mathbf{r}}')$$

Electromagnetic waves from time harmonic sources –
continued:

Lowest order contribution; dipole radiation :

Define dipole moment at frequency ω :

$$\mathbf{p}(\omega) \equiv \int d^3r \, \mathbf{r} \tilde{\rho}(\mathbf{r},\omega) = -\frac{1}{i\omega} \int d^3r \, \tilde{\mathbf{J}}(\mathbf{r},\omega)$$

$$\tilde{\mathbf{A}}(\mathbf{r},\omega) = -\frac{i\mu_0\omega}{4\pi} \mathbf{p}(\omega) \frac{e^{ikr}}{r}$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\frac{i}{4\pi\omega\varepsilon_0} \mathbf{p}(\omega) \cdot \hat{\mathbf{r}} \left(1 + \frac{i}{kr}\right) \frac{e^{ikr}}{r}$$

Note: in this case we have only assumed the throughout
the extent of the source *kr'<<1.*

Electromagnetic waves from time harmonic sources –
continued:

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega) = -\nabla \widetilde{\Phi}(\mathbf{r},\omega) + i\omega \widetilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{e^{ikr}}{r} \left(k^2 \left((\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}} \right) + \left(\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}(\omega)) - \mathbf{p}(\omega)}{r^2} \right)(1 - ikr) \right)$$

$$\widetilde{\mathbf{B}}(\mathbf{r},\omega) = \nabla \times \widetilde{\mathbf{A}}(\mathbf{r},\omega)$$

$$= \frac{1}{4\pi\varepsilon_0 c^2} \frac{e^{ikr}}{r} k^2 (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \left(1 - \frac{1}{ikr} \right)$$

Power radiated for $kr \gg 1$:

$$\frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S} \rangle_{avg} = \frac{r^2 \hat{\mathbf{r}}}{2\mu_0} \hat{\mathbf{r}} \cdot \Re \left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{B}}^*(\mathbf{r},\omega) \right)$$

$$= \frac{c^2 k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \left| (\hat{\mathbf{r}} \times \mathbf{p}(\omega)) \times \hat{\mathbf{r}} \right|^2$$