Electrodynamics – PHY712 – Lecture 22

Electromagnetic waves in waveguides – Section 8.4 in Jackson's text

For an "ideal" conductor, $\sigma \to \infty$, so that the electric and magnetic fields are confined to the surface of the conductor. Because of the field continuity conditions at the surface of the conductor, this means that, $\mathbf{H}_{\mathrm{tangential}} \neq 0$ (because there can be a surface current), $\mathbf{D}_{\mathrm{normal}} \neq 0$ (because there can be a surface charge), but $\mathbf{B}_{\mathrm{normal}} = 0$ and $\mathbf{E}_{\mathrm{tangential}} = 0$.

Suppose we construct a wave guide from an "ideal" conductor, designating $\hat{\mathbf{z}}$ as the propagation direction. We will express the fields in terms of $\mathbf{B}$ and $\mathbf{E}$ and assume that within the wave guide the permittivity ϵ and permeablity μ parameters are isotropic and real. We will assume that the fields take the form:

$$\mathbf{E} = \mathbf{E}(x, y)\mathrm{e}^{ikz - i\omega t} \quad \text{and} \quad \mathbf{B} = \mathbf{B}(x, y)\mathrm{e}^{ikz - i\omega t} \tag{1}$$

inside the pipe, where now k and ε are assumed to be real.

Electromagnetic waves in waveguides – continued

Assuming that there are no sources inside the pipe, the fields there must satisfy Maxwell's equations (8.16) which expand to the following :

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0. \tag{2}$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0. \tag{3}$$

$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x. \tag{4}$$

$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y. \tag{5}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z. \tag{6}$$

$$\frac{\partial B_z}{\partial y} - ikB_y = -i\mu\varepsilon\omega E_x. \tag{7}$$

$$ikB_x - \frac{\partial B_z}{\partial x} = -i\mu\varepsilon\omega E_y. \tag{8}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\mu\varepsilon\omega E_z. \tag{9}$$

Electromagnetic waves in waveguides – continued

Combining Faraday's Law and Ampere's Law, we find that each field component must satisfy a two-dimensional Helmholz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k^2 + \mu\varepsilon\omega^2\right) E_x(x,y) = 0, \tag{10}$$

with similar expressions for each of the other field components. For the rectangular wave guide discussed in Section 8.4 of your text a solution for a TE mode can have:

$$E_z(x,y) \equiv 0 \quad \text{and} \quad B_z(x,y) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \tag{11}$$

with $k^2 \equiv k^2_{mn} = \mu\varepsilon\omega^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$. From this result and Maxwell's equations, we can determine the other field components. For example Eqs. (4-5) simplify to

$$B_x = -\frac{k}{\omega}E_y \quad \text{and} \quad B_y = \frac{k}{\omega}E_x. \tag{12}$$

Electromagnetic waves in waveguides – continued

These results can be used in Eqs. (7-8) to solve for the fields E_x and E_y and B_x and B_y:

$$E_x = \frac{\omega}{k} B_y = \frac{-i\omega}{k^2 - \mu\varepsilon\omega^2} \frac{\partial B_z}{\partial y} = \frac{-i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \frac{n\pi}{b} B_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \tag{13}$$

and

$$E_y = -\frac{\omega}{k} B_x = \frac{i\omega}{k^2 - \mu\varepsilon\omega^2} \frac{\partial B_z}{\partial x} = \frac{i\omega}{\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \frac{m\pi}{a} B_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right). \tag{14}$$

One can check this result to show that these results satisfy the boundary conditions. For example, $\mathbf{E}_{\text{tangential}} = 0$ is satisfied since $E_x(x,0) = E_x(x,b) = 0$ and $E_y(0,y) = E_y(a,y) = 0$. This was made possible choosing $\nabla B_z\rfloor_{\text{surface}} \cdot \hat{\mathbf{n}} = 0$, where $\hat{\mathbf{n}}$ denotes a unit normal vector pointing out of the wave guide surface.