PHY 712 Electrodynamics
11-11:50 AM MWF Olin 107

Plan for Lecture 20:

Continue reading Chapter 7

A. Frequency dependence of dielectric materials – Drude model

B. Kramer's Kronig transformation

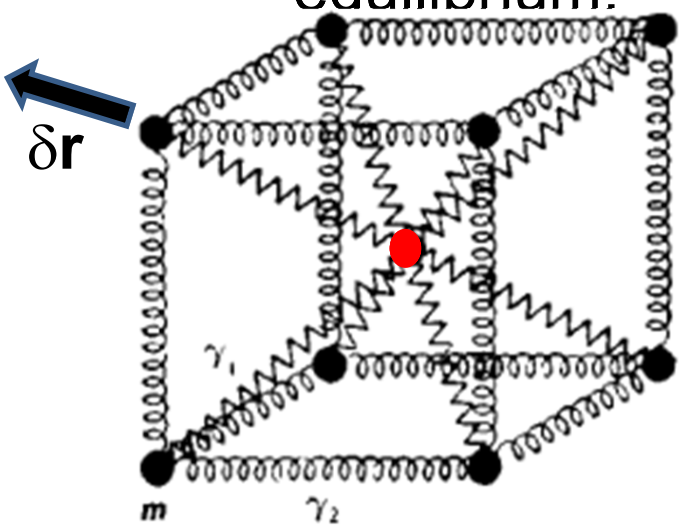
13	02-15(Fri)	Chap. 5	Magnetic dipole fields	Exam
14	02-18(Mon)	Chap. 5	Permeable media	Exam
15	02-20(Wed)	Chap. 5	Magnetic susceptibility and permeability	Exam
16	02-22(Fri)	Chap. 6	Maxwell's equations	Exam
17	02-25(Mon)	Chap. 6	Poynting Vector	#11
18	02-27(Wed)	Chap. 7	Reflectance and transmittance of electromagnetic plane waves	#12
19	02-28(Thur)	Chap. 7	Anisotropic media	#13
20	03-01(Fri)	Chap. 7	Dielectric models; Kramers-Kronig Relations	#14
21	03-04(Mon)			
22	03-06(Wed)			
23	03-07(Thur)			
24	03-08(Fri)			
	03-11(Mon)	*Spring Break*		
	03-13(Mon)	*Spring Break*		
	03-15(Fri)	*Spring Break*		
	03-18(Mon)	*APS Meeting*		Exam
	03-20(Wed)	*APS Meeting*		Exam
	03-22(Fri)	*APS Meeting*		Exam

Paul Karl Ludwig Drude   1863-1906

http://photos.aip.org/history/Thumbnails/drude_paul_a1.jpg

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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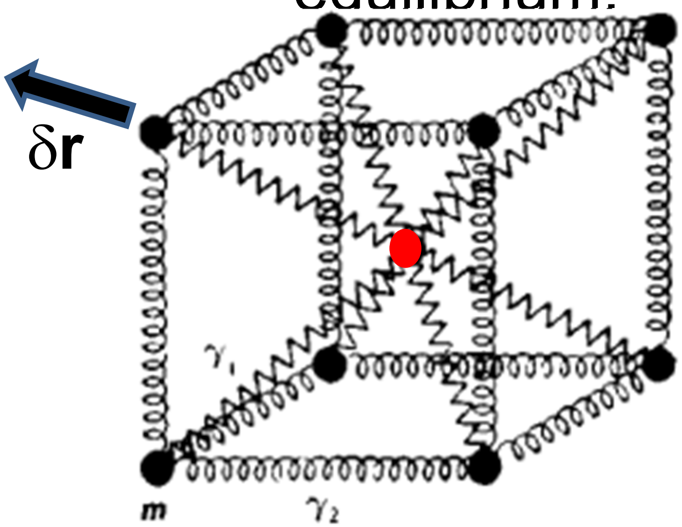
$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

Note that:

- $\gamma > 0$ represents dissipation of energy.
- ω_0 represents the natural frequency of the vibration; $\omega_0=0$ would represent a free (unbound) particle

Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2 \delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

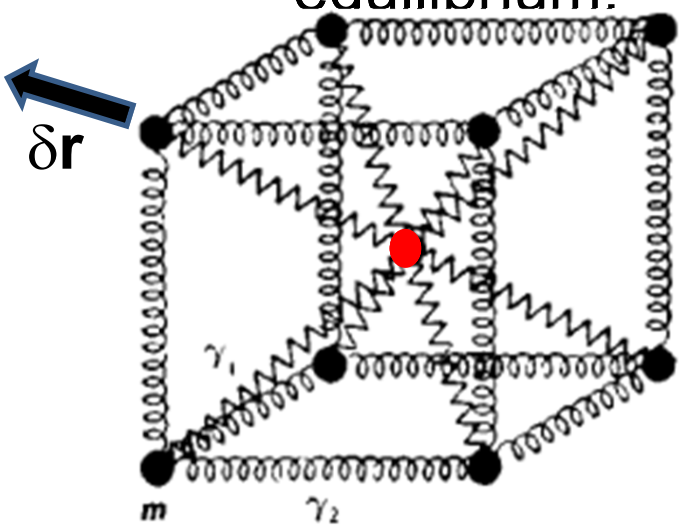
$$\text{For} \quad \delta\mathbf{r} \equiv \delta\mathbf{r}_0 e^{-i\omega t}, \qquad \delta\mathbf{r}_0 = \frac{q\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

Induced dipole:

$$\mathbf{p} = q\ \delta\mathbf{r} = \frac{q^2\mathbf{E}_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}e^{-i\omega t}$$

Drude model:

Vibration of particle of charge q and mass m near equilibrium:



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$$m\delta\ddot{\mathbf{r}} = q\mathbf{E}_0 e^{-i\omega t} - m\omega_0^2\delta\mathbf{r} - m\gamma\delta\dot{\mathbf{r}}$$

Displacement field:

$$\mathbf{D} = \varepsilon\mathbf{E} \quad = \quad \varepsilon_0\mathbf{E} + \mathbf{P}$$

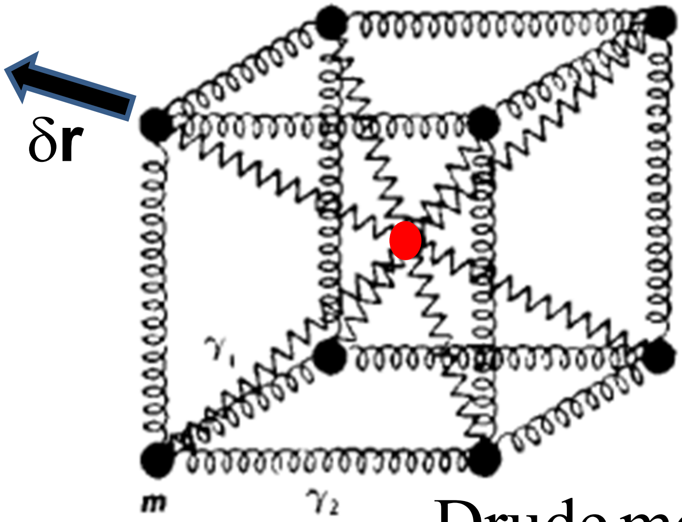
$$\mathbf{P} = \sum_i \mathbf{p}_i \delta^3(\mathbf{r} - \mathbf{r}_i) \approx N\sum_i f_i\,\mathbf{p}_i$$

$N \equiv$ number dipole/volume

$f_i \equiv$ fraction of type i dipoles

Drude model:

Vibration of particle of charge *q* and mass *m* near equilibrium:



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Drude model expression for permittivity:

$$\mathbf{D} = \varepsilon\mathbf{E} \quad = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\mathbf{E} + N\sum_i f_i\,\mathbf{p}_i$$

$$\mathbf{p}_i = q_i\,\delta\mathbf{r} = \frac{q_i^{\,2}\mathbf{E}_0}{m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}e^{-i\omega t}$$

$$\varepsilon\mathbf{E} \quad = \varepsilon_0\mathbf{E}_0 e^{-i\omega t}\left(1 + N\sum_i f_i\frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}\right)$$

Drude model dielectric function:

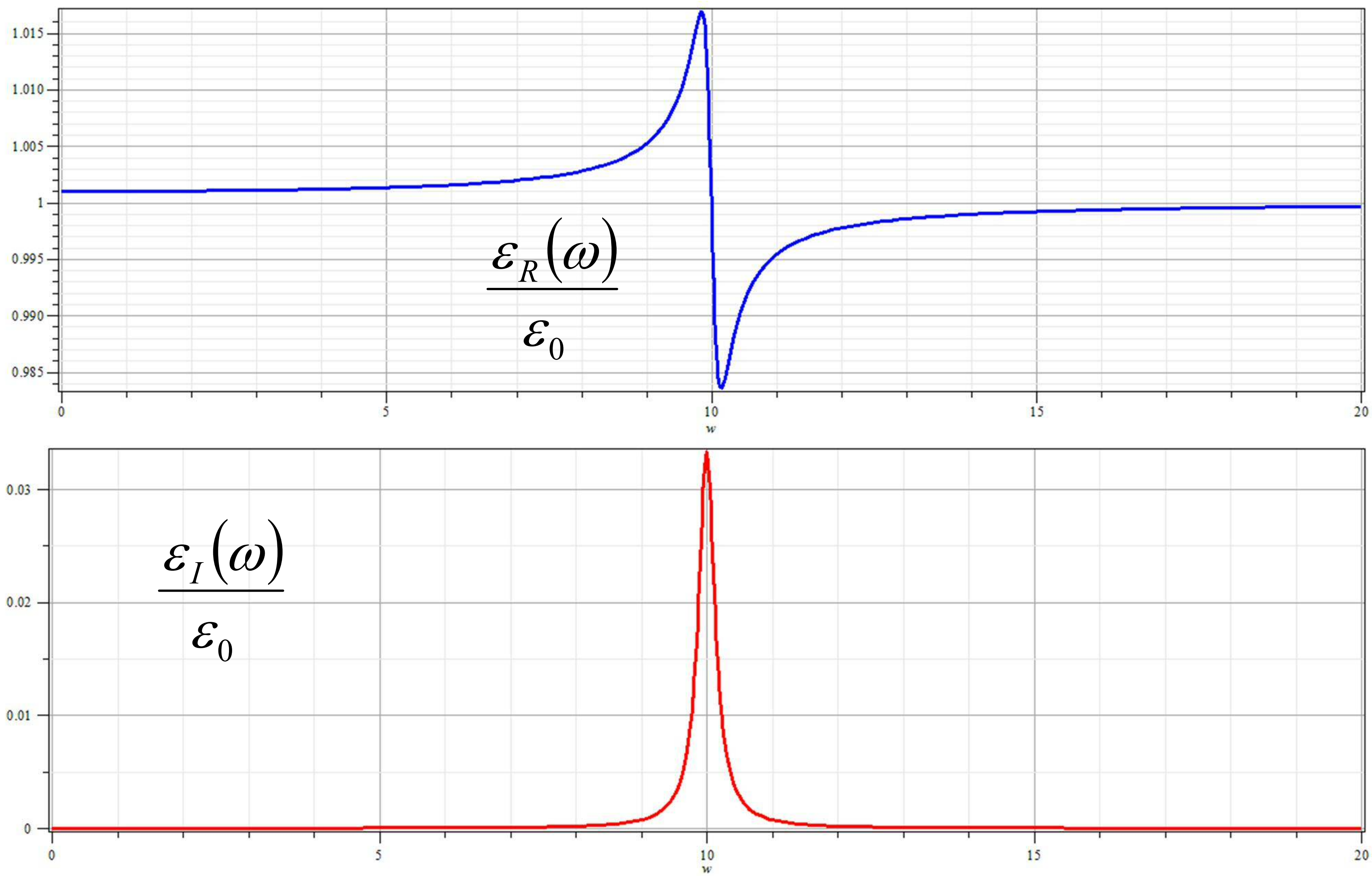
$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

$$= \frac{\varepsilon_R(\omega)}{\varepsilon_0} + i\frac{\varepsilon_I(\omega)}{\varepsilon_0}$$

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{\omega_i^2 - \omega^2}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^{\,2}}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{\omega\gamma_i}{\left(\omega_i^2 - \omega^2\right)^2 + \omega^2\gamma_i^{\,2}}$$

Drude model dielectric function:



Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega >> \omega_i$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \approx 1 - \frac{1}{\omega^2}\left(N\sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \right)$$

$$\equiv 1 - \frac{\omega_P^2}{\omega^2}$$

Drude model dielectric function – some analytic properties:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

For $\omega_0 = 0$ (representing a free particle of charge q_0, mass m_0

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_{i>0} f_i \frac{q_i^{\ 2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i} + iNf_0 \frac{q_0^{\ 2}}{\varepsilon_0 m_0} \frac{1}{\omega(\gamma_0 - i\omega)}$$

$$\equiv \frac{\varepsilon_b(\omega)}{\varepsilon_0} + i\frac{\sigma(\omega)}{\varepsilon_0 \omega}$$

Some details:

$$\mathbf{D} = \varepsilon_b \mathbf{E} \qquad\qquad \mathbf{J} = \sigma \mathbf{E}$$

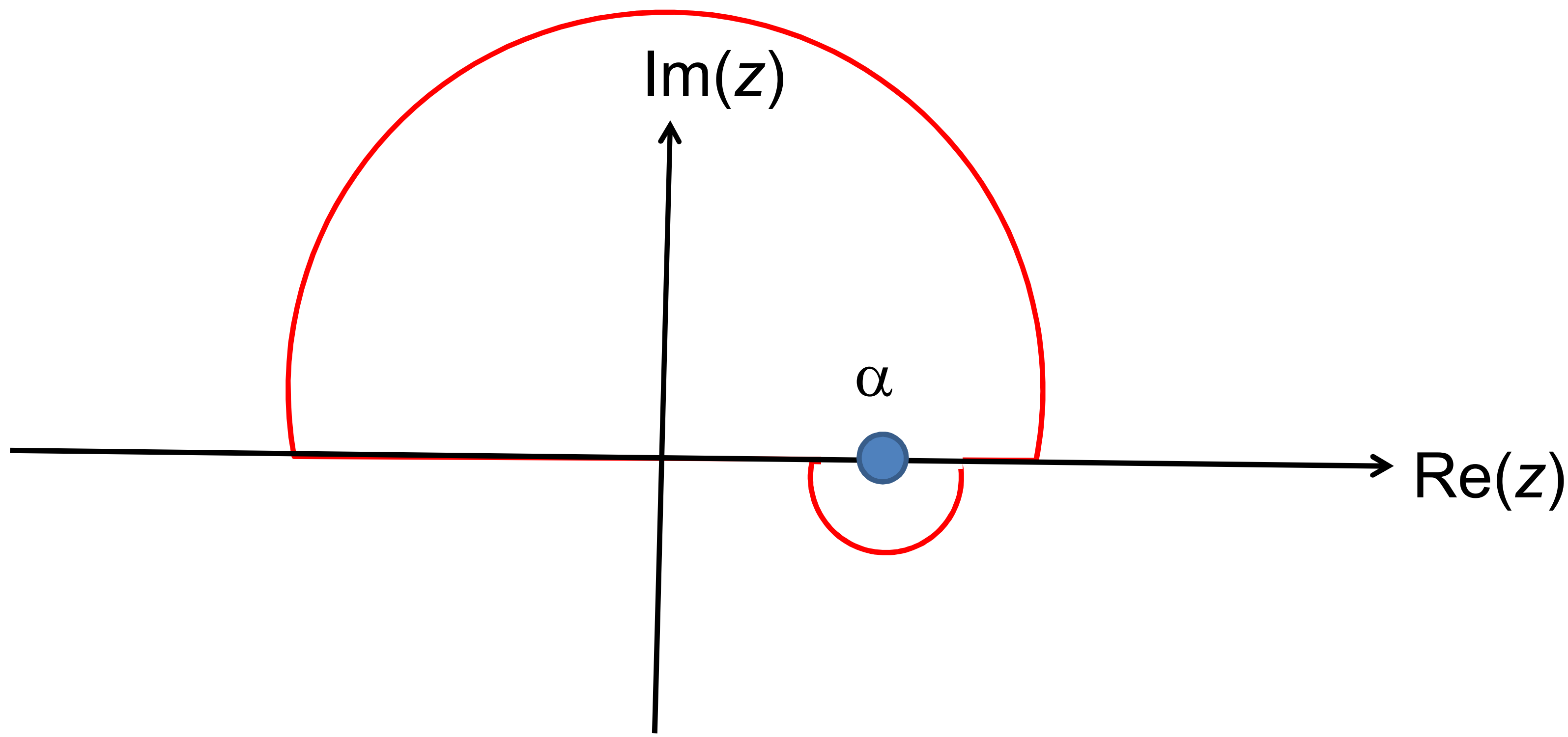
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = (\sigma - i\omega\varepsilon_b)\mathbf{E} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} = -i\omega\left(\varepsilon_b + \frac{i\sigma}{\omega}\right)\mathbf{E}$$

$$\Rightarrow \sigma(\omega) = Nf_0 \frac{q_0^{\ 2}}{m_0} \frac{1}{(\gamma_0 - i\omega)}$$

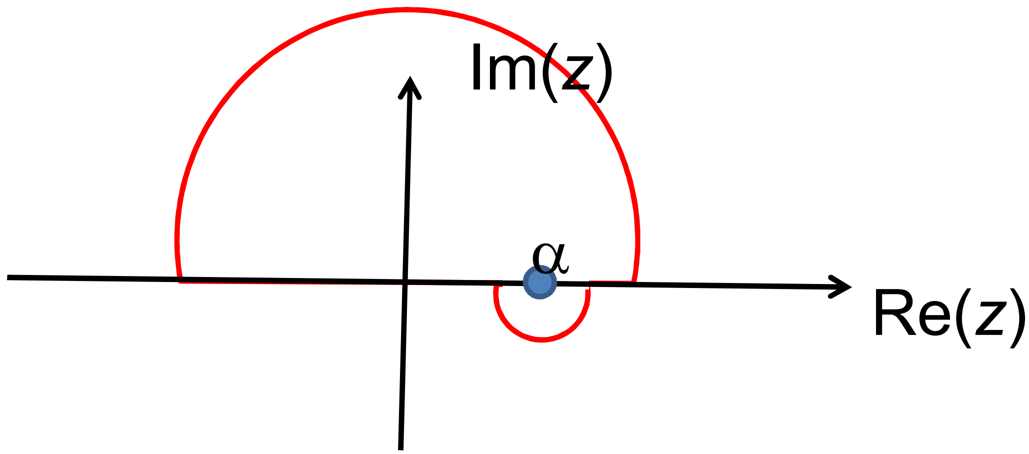
Analytic properties of the dielectric function (in the Drude model or from "first principles" -- Kramers-Kronig transform

Consider Cauchy's integral formula for an analytic function $f(z)$:

$$\oint dz\ f(z) = 0 \qquad\qquad f(\alpha) = \frac{1}{2\pi i} \oint_{includes\,\alpha} dz\, \frac{f(z)}{z-\alpha}$$



Kramers-Kronig transform -- continued



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$$f(\alpha)=\frac{1}{2\pi i}\oint_{includes\,\alpha} dz\,\frac{f(z)}{z-\alpha} = \frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}+\int_{rest} dz\,\frac{f(z)}{z-\alpha}\right)$$

$$f(\alpha)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}=\frac{1}{2\pi i}P\int_{-\infty}^{\infty} dz_R\,\frac{f(z_R)}{z_R-\alpha}+\frac{1}{2}f(\alpha)$$

Kramers-Kronig transform -- continued

$$f(\alpha) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f(z_R)}{z_R - \alpha} + \frac{1}{2} f(\alpha)$$

Suppose $f(z_R) = f_R(z_R) + i f_I(z_R)$:

$$\Rightarrow \frac{1}{2}\left(f_R(\alpha) + i f_I(\alpha)\right) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R) + i f_I(z_R)}{z_R - \alpha}$$

$$\Rightarrow f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_I(z_R)}{z_R - \alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \frac{f_R(z_R)}{z_R - \alpha}$$

Kramers-Kronig transform -- continued

$$f_R(\alpha) = \frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \ \frac{f_I(z_R)}{z_R - \alpha}$$

$$f_I(\alpha) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} dz_R \ \frac{f_R(z_R)}{z_R - \alpha}$$

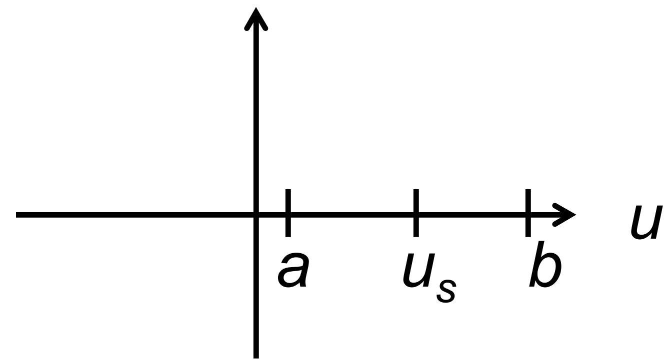
This Kramers-Kronig transform is useful for the dielectric function

when $\quad f(z_R) \Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} - 1$

Must show that: 1. $f(z)$ is analytic for $z_I \geq 0$

2. $f(z)$ vanishes for $z \to \infty$

Some practical considerations



Principal parts integration :

$$P\int_a^b du\; g(u) = \lim_{\upsilon\to 0}\left(\int_a^{u_s-\upsilon} du\; g(u) + \int_{u_s+\upsilon}^{b} du\; g(u)\right)$$

Example:

$$P\int_a^b du\;\frac{1}{u\text{-}u_s} = \lim_{\upsilon\to 0}\left(\int_a^{u_s-\upsilon} du\,\frac{1}{u\text{-}u_s} + \int_{u_s+\upsilon}^{b} du\,\frac{1}{u\text{-}u_s}\right)$$

$$= \lim_{\upsilon\to 0}\left(\ln\left(\frac{\upsilon}{u_s-a}\right) + \ln\left(\frac{b-u_s}{\upsilon}\right)\right) = \ln\left(\frac{b-u_s}{u_s-a}\right)$$

More practical considerations

For dielectric function $\varepsilon(\omega)$:

$$\varepsilon(-\omega) = \varepsilon^*(\omega)$$

$$\Rightarrow \varepsilon_R(-\omega) = \varepsilon_R(\omega)$$

$$\Rightarrow \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Analytic properties the dielectric function which justify the treatment of $f(z) \Rightarrow \frac{\varepsilon(z)}{\varepsilon_0} - 1$

Must show that: 1. $f(z)$ is analytic for $z_I \geq 0$

2. $f(z)$ vanishes for $z \to \infty$ (for $z_I \geq 0$)

Analysis for Drude model dielectric function:

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

Let $f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i}\frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$

For $|z| >> \omega_i$

$$f(z) \approx -\frac{1}{z^2}\left(N\sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i}\right) \Rightarrow \text{vanishes at large } z$$

Analysis for Drude model dielectric function – continued --
Analytic properties:

$$f(z) = \frac{\varepsilon(z)}{\varepsilon_0} - 1 = N \sum_i f_i \frac{q_i^{\,2}}{\varepsilon_0 m_i} \frac{1}{\omega_i^2 - z^2 - iz\gamma_i}$$

$f(z)$ has poles z_P at $\omega_i^2 - z_P^{\,2} - iz_P\gamma_i = 0$

$$z_P = -i\frac{\gamma_i}{2} \pm \sqrt{\omega_i^2 - \left(\frac{\gamma_i}{2}\right)^2}$$

Note that $\Im(z_P) \leq 0 \;\Rightarrow\; f(z)$ is analytic for $\Im(z_P) > 0$

Kramers-Kronig transform – for dielectric function:

$$\frac{\varepsilon_R(\omega)}{\varepsilon_0} - 1 = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\varepsilon_I(\omega')}{\varepsilon_0} \frac{1}{\omega' - \omega}$$

$$\frac{\varepsilon_I(\omega)}{\varepsilon_0} = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \left(\frac{\varepsilon_R(\omega')}{\varepsilon_0} - 1 \right) \frac{1}{\omega' - \omega}$$

$$\text{with} \quad \varepsilon_R(-\omega) = \varepsilon_R(\omega); \quad \varepsilon_I(-\omega) = -\varepsilon_I(\omega)$$

Further comments on analytic behavior of dielectric function

"Causal" relationship between $\mathbf{E}$ and $\mathbf{D}$ fields:

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \left\{ \mathbf{E}(\mathbf{r},t) + \int_0^\infty d\tau \; G(\tau) \mathbf{E}(\mathbf{r},t-\tau) \right\}$$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} - 1 = \int_0^\infty d\tau \; G(\tau) e^{i\omega\tau}$$