PHY 712 Electrodynamics
11-11:50 AM MWF Olin 107

Plan for Lecture 17:

Finish reading Chapter 6

A. Some details of Lienard-Wiechert results

B. Energy associated with electromagnetic fields

C. Comment on Exam 1

10	02-08(Fri)	Chap. 4	Microscopic and macroscopic polarizability	
11	02-11(Mon)	Chap. 5	Magnetostatics	#10
12	02-13(Wed)	Chap. 5	Magnetostatic fields	
13	02-15(Fri)	Chap. 5	Magnetic dipole fields	Exam
14	02-18(Mon)	Chap. 5	Permeable media	Exam
15	02-20(Wed)	Chap. 5	Magnetic susceptibility and permeability	Exam
16	02-22(Fri)	Chap. 6	Maxwell's equations	Exam
17	02-25(Mon)	Chap. 6	Poynting Vector	#11
18	02-27(Wed)			
19	03-01(Fri)			
20	03-04(Mon)			
21	03-06(Wed)			
22	03-08(Fri)			
	03-11(Mon)	*Spring Break*		
	03-13(Mon)	*Spring Break*		
	03-15(Fri)	*Spring Break*		
	03-18(Mon)	*APS Meeting*		Exam
	03-20(Wed)	*APS Meeting*		Exam
	03-22(Fri)	*APS Meeting*		Exam

➔Please send me your schedules so we can plan for 3 make-up classes.

Comment on Lienard-Wiechert potential results

$$\int_{-\infty}^{\infty} dt' \; f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

Note that for any function $F(x)$:

$$\int_{-\infty}^{\infty} F(x)\delta(x - x_0)dx = F(x_0)$$

Now consider a function $p(x)$, for which $p(x_i) = 0$ for $i = 1,2,\cdots$

$$\int_{-\infty}^{\infty} F(x)\delta(p(x))dx = \int_{-\infty}^{\infty} F(x)\left(\sum_i \delta\left(\left(x - x_i\right)\left.\frac{dp}{dx}\right|_{x_i}\right)\right)dx$$

$$= \sum_i \frac{F(x_i)}{\left|\frac{dp}{dx}\right|_{x_i}}$$

Comment on Lienard-Wiechert potential results -- continued

$$\int_{-\infty}^{\infty} dt'\; f(t')\delta\left(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)\right) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot(\mathbf{r}-\mathbf{R}_q(t_r))}{c|\mathbf{r}-\mathbf{R}_q(t_r)|}},$$

where the "retarded time" is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}.$$

In this case we have: $\displaystyle\int_{-\infty}^{\infty} f(t')\delta\big(p(t')\big)dt' = \frac{f(t_r)}{\left|1 - \frac{\dot{\mathbf{R}}_q(t_r)\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t_r)\right|}\right|}$

where: $\displaystyle p(t') \equiv t' - \left(t - \frac{\left|\mathbf{r}-\mathbf{R}_q(t')\right|}{c}\right)$

$$\frac{\partial p(t')}{\partial t'} = 1 - \frac{\frac{\partial \mathbf{R}_q(t')}{\partial t'}\cdot\left(\mathbf{r}-\mathbf{R}_q(t')\right)}{c\left|\mathbf{r}-\mathbf{R}_q(t')\right|}$$

Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Energy analysis of electromagnetic fields and sources

Rate of work done on source $\mathbf{J}(\mathbf{r},t)$ by electromagnetic field:

$$\frac{dW}{dt} \equiv \frac{dE_{mech}}{dt} = \int d^3r \;\; \mathbf{E} \cdot \mathbf{J}$$

Expressing source current in terms of fields it produces:

$$\frac{dW}{dt} = \int d^3r \;\; \mathbf{E} \cdot \left(\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} \right)$$

Energy analysis of electromagnetic fields and sources -
- continued

$$\frac{dW}{dt} = \int d^3r\ \mathbf{E}\cdot\mathbf{J} = \int d^3r\ \mathbf{E}\cdot\left(\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\right)$$

$$= -\int d^3r\ \left(\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right) + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)$$

Let $\mathbf{S} \equiv \mathbf{E}\times\mathbf{H}$ "Poynting vector"

$$u \equiv \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}\right) \quad \text{energy density}$$

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{E}\cdot\mathbf{J}$$

Energy analysis of electromagnetic fields and sources -
- continued

$$\frac{dE_{mech}}{dt} \equiv \int d^3r \;\; \mathbf{E} \cdot \mathbf{J}$$

$$\text{Electromagnetic energy density:} \qquad u \equiv \frac{1}{2}\left(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}\right)$$

$$E_{field} \equiv \int d^3r \;\; u(\mathbf{r}, t)$$

$$\text{Poynting vector:} \qquad \mathbf{S} \equiv \mathbf{E} \times \mathbf{H}$$

$$\text{From the previous energy analysis:} \; \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J}$$

$$\Rightarrow \frac{dE_{mech}}{dt} + \frac{dE_{field}}{dt} = -\int d^3r \;\; \nabla \cdot \mathbf{S}(\mathbf{r}, t) = -\oint d^2r \; \hat{\mathbf{r}} \cdot \mathbf{S}(\mathbf{r}, t)$$

Momentum analysis of electromagnetic fields and sources

$$\frac{d\mathbf{P}_{mech}}{dt} \equiv \int d^3r \ \left(\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right)$$

Follows by analogy with Lorentz force:

$$F = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

$$\mathbf{P}_{field} = \varepsilon_0 \int d^3r \ \left(\mathbf{E}\times\mathbf{B}\right)$$

Expression for vacuum fields:

$$\left(\frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{field}}{dt}\right)_i = \sum_j \int d^3r \ \frac{\partial T_{ij}}{\partial r_j}$$

Maxwell stress tensor:

$$T_{ij} \equiv \varepsilon_0 \left(E_i E_j + c^2 B_i B_j - \delta_{ij} \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}\right)\right)$$

Comment on treatment of time-harmonic fields

Fourier transformation in time domain:

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \widetilde{\mathbf{E}}(\mathbf{r},\omega)\, e^{-i\omega t}$$

$$\widetilde{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt\, \mathbf{E}(\mathbf{r},t)\, e^{i\omega t}$$

Note that $\mathbf{E}(\mathbf{r},t)$ is real $\Rightarrow \widetilde{\mathbf{E}}(\mathbf{r},\omega) = \widetilde{\mathbf{E}}^*(\mathbf{r},-\omega)$

These relations and the notion of the superposition principle, lead to the common treatment:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Coulomb's law:	$\nabla \cdot \mathbf{D} = \rho_{free}$	$\nabla \cdot \widetilde{\mathbf{D}} = \widetilde{\rho}_{free}$
Ampere-Maxwell's law:	$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$	$\nabla \times \widetilde{\mathbf{H}} + i\omega\widetilde{\mathbf{D}} = \widetilde{\mathbf{J}}_{free}$
Faraday's law:	$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$	$\nabla \times \widetilde{\mathbf{E}} - i\omega\widetilde{\mathbf{B}} = 0$
No magnetic monopoles:	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \widetilde{\mathbf{B}} = 0$

Note -- in all of these, the real part is taken at the end of the calculation.

Comment on treatment of time-harmonic fields -- continued

Equations for time harmonic fields:

$$\mathbf{E}(\mathbf{r},t) = \Re\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}\right) \equiv \frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

Poynting vector: $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)$

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega)e^{i\omega t}\right) \times \left(\widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-i\omega t} + \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{i\omega t}\right)$$

$$= \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}^*(\mathbf{r},\omega) + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}(\mathbf{r},\omega)\right)$$

$$+ \frac{1}{4}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}(\mathbf{r},\omega)e^{-2i\omega t} + \widetilde{\mathbf{E}}^*(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)e^{2i\omega t}\right)$$

$$\left\langle \mathbf{S}(\mathbf{r},t) \right\rangle_{t\,\text{avg}} = \Re\left(\frac{1}{2}\left(\widetilde{\mathbf{E}}(\mathbf{r},\omega) \times \widetilde{\mathbf{H}}^*(\mathbf{r},\omega)\right)\right)$$