PHY 712 Electrodynamics
11-11:50 AM MWF Olin 107

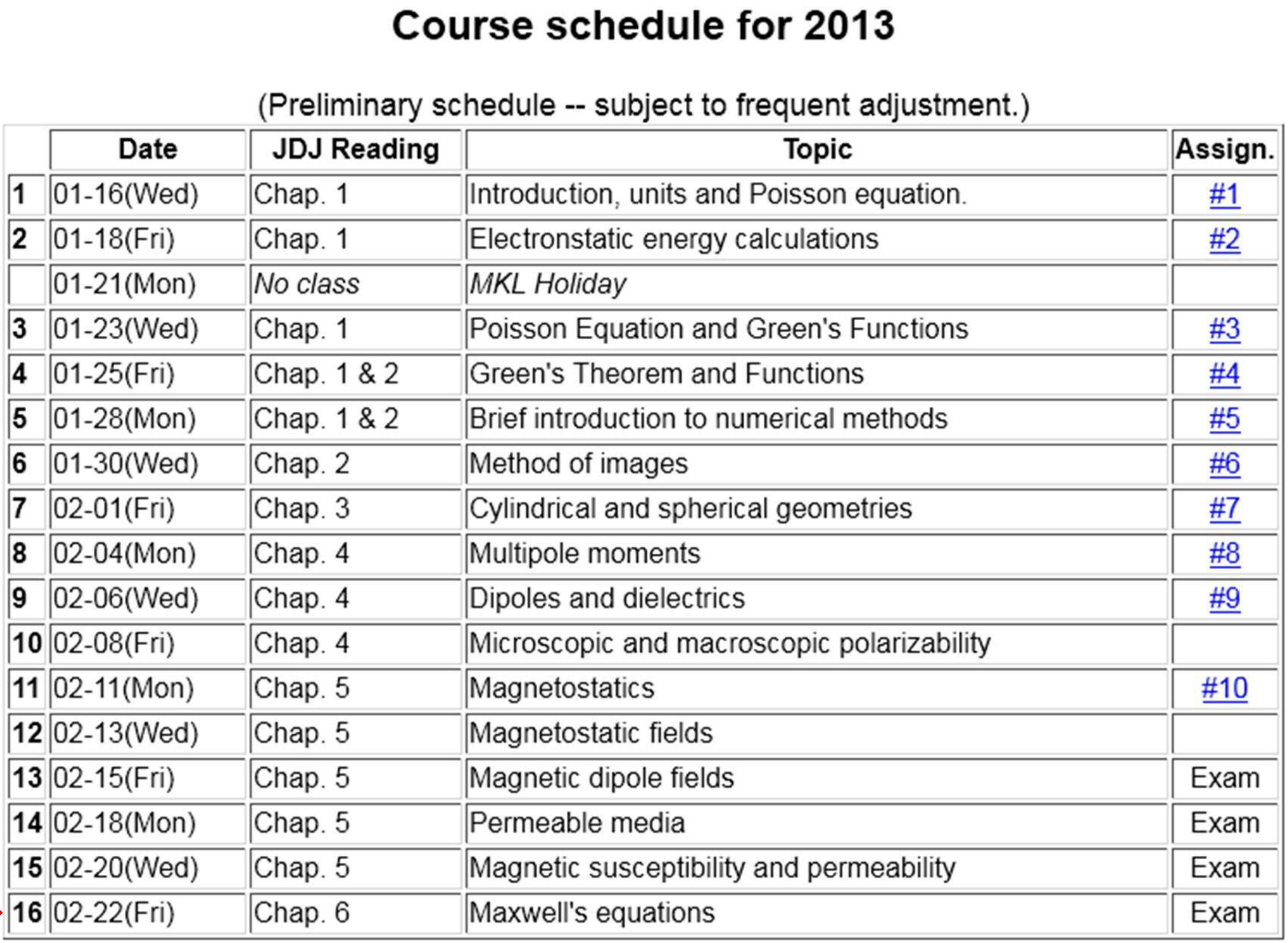
Plan for Lecture 16:

Continue reading Chapter 6

A. **Maxwell's equations**

B. **Vector and scalar potential representations**

C. **Green's function solution Maxwell's equations**



Course schedule for 2013

(Preliminary schedule -- subject to frequent adjustment.)

	Date	JDJ Reading	Topic	Assign.
1	01-16(Wed)	Chap. 1	Introduction, units and Poisson equation.	#1
2	01-18(Fri)	Chap. 1	Electronstatic energy calculations	#2
	01-21(Mon)	*No class*	*MKL Holiday*	
3	01-23(Wed)	Chap. 1	Poisson Equation and Green's Functions	#3
4	01-25(Fri)	Chap. 1 & 2	Green's Theorem and Functions	#4
5	01-28(Mon)	Chap. 1 & 2	Brief introduction to numerical methods	#5
6	01-30(Wed)	Chap. 2	Method of images	#6
7	02-01(Fri)	Chap. 3	Cylindrical and spherical geometries	#7
8	02-04(Mon)	Chap. 4	Multipole moments	#8
9	02-06(Wed)	Chap. 4	Dipoles and dielectrics	#9
10	02-08(Fri)	Chap. 4	Microscopic and macroscopic polarizability	
11	02-11(Mon)	Chap. 5	Magnetostatics	#10
12	02-13(Wed)	Chap. 5	Magnetostatic fields	
13	02-15(Fri)	Chap. 5	Magnetic dipole fields	Exam
14	02-18(Mon)	Chap. 5	Permeable media	Exam
15	02-20(Wed)	Chap. 5	Magnetic susceptibility and permeability	Exam
16	02-22(Fri)	Chap. 6	Maxwell's equations	Exam

Maxwell's equations

Coulomb's law: $\nabla \cdot \mathbf{D} = \rho_{free}$

Ampere-Maxwell's law: $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}_{free}$

Faraday's law: $\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$

No magnetic monopoles: $\nabla \cdot \mathbf{B} = 0$

Maxwell's equations

$$\text{Microscopic or vacuum form} \quad (\mathbf{P}=0;\ \mathbf{M}=0):$$

$$\text{Coulomb's law:} \qquad \nabla\cdot\mathbf{E}=\rho/\varepsilon_0$$

$$\text{Ampere-Maxwell's law:} \qquad \nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mu_0\mathbf{J}$$

$$\text{Faraday's law:} \qquad \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0$$

$$\text{No magnetic monopoles:} \qquad \nabla\cdot\mathbf{B}=0$$

$$\Rightarrow c^2=\frac{1}{\varepsilon_0\mu_0}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials

$$\nabla \cdot \mathbf{B} = 0 \qquad \Rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \Rightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

$$\text{or} \quad \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Analysis of the scalar and vector potential equations:

$$-\nabla^2\Phi - \frac{\partial\left(\nabla\cdot\mathbf{A}\right)}{\partial t} = \rho/\varepsilon_0$$

$$\nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{1}{c^2}\left(\frac{\partial\left(\nabla\Phi\right)}{\partial t} + \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mu_0\mathbf{J}$$

Lorentz gauge form -- require $\nabla\cdot\mathbf{A}_L + \frac{1}{c^2}\frac{\partial\Phi_L}{\partial t} = 0$

$$-\nabla^2\Phi_L + \frac{1}{c^2}\frac{\partial^2\Phi_L}{\partial t^2} = \rho/\varepsilon_0$$

$$-\nabla^2\mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = \mu_0\mathbf{J}$$

Formulation of Maxwell's equations in terms of vector and scalar potentials -- continued

Lorentz gauge form -- require $\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \Phi_L}{\partial t} = 0$

$$-\nabla^2 \Phi_L + \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = \rho / \varepsilon_0$$

$$-\nabla^2 \mathbf{A}_L + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = \mu_0 \mathbf{J}$$

Alternate potentials: $\mathbf{A}'_L = \mathbf{A}_L + \nabla\Lambda$ and $\Phi'_L = \Phi_L - \frac{\partial \Lambda}{\partial t}$

Yields same physics provided that: $\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0$

Solution of Maxwell's equations in the Lorentz gauge

$$\nabla^2 \Phi_L - \frac{1}{c^2}\frac{\partial^2 \Phi_L}{\partial t^2} = -\rho/\varepsilon_0$$

$$\nabla^2 \mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{J}$$

Consider the general form of the 3-dimensional wave equation:

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f$$

$\Psi(\mathbf{r},t) \Rightarrow$ wave field $\quad f(\mathbf{r},t) \Rightarrow$ source

Solution of Maxwell’s equations in the Lorentz gauge -- continued

$$\nabla^2\Psi(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2\Psi(\mathbf{r},t)}{\partial t^2}=-4\pi f(\mathbf{r},t)$$

Green's function :

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t')=-4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t)=\Psi_{f=0}(\mathbf{r},t)+\int d^3r'\int dt' G(\mathbf{r},t;\mathbf{r}',t')f(\mathbf{r}',t')$$

Solution of Maxwell’s equations in the Lorentz gauge -- continued

Determination of the form for the Green's function :

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

For the case of isotropic boundary values at infinity :

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)$$

Formal solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t) = \Psi_{f=0}(\mathbf{r},t) +$$

$$\int d^3r' \int dt' \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t' - \left(t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right) f(\mathbf{r}',t')$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

Analysis of the Green's function:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Fourier analysis in the time domain -- note that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}$$

Define:

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')}\, \widetilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$\Rightarrow \quad \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution of Maxwell’s equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

For the case of isotropic boundary values at infinity:

$$\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = \widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)$$

Further assuming that $\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)$ is isotropic in $|\mathbf{r}-\mathbf{r}'| \equiv R$:

$$\left(\frac{1}{R}\frac{d^2}{dR^2}R + \frac{\omega^2}{c^2}\right)\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$$

Solution: $\widetilde{G}(\mathbf{r},\mathbf{r}',\omega) = \frac{1}{R}e^{\pm i\omega R/c}$

Solution of Maxwell’s equations in the Lorentz gauge -- continued

Analysis of the Green's function (continued):

$$\widetilde{G}(\mathbf{r},\mathbf{r}',\omega)=\frac{1}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega(t-t')}\,\widetilde{G}(\mathbf{r},\mathbf{r}',\omega)$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega(t-t')}\frac{1}{|\mathbf{r}-\mathbf{r}'|}e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}$$

$$=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{i\omega(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c)}\right)$$

$$=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(t-t'\pm|\mathbf{r}-\mathbf{r}'|/c)=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta(t'-t\mp|\mathbf{r}-\mathbf{r}'|/c)$$

Solution of Maxwell's equations in the Lorentz gauge -- continued

$$G(\mathbf{r},t;\mathbf{r}',t')=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t\pm|\mathbf{r}-\mathbf{r}'|/c\right)\right)$$

Solution for field $\Psi(\mathbf{r},t)$:

$$\Psi(\mathbf{r},t)=\Psi_{f=0}(\mathbf{r},t)+$$

$$\int d^3r'\int dt'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-\left(t-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right)\right)f(\mathbf{r}',t')$$