

**PHY 712 Electrodynamics**  
**11-11:50 AM MWF Olin 107**

**Plan for Lecture 14:**

**Continue reading Chapter 5**

**A. Macroscopic magnetization density  $M$**

**B.  $H$  field and its relation to  $B$**

**C. Magnetic boundary values**

## Course schedule for 2013

(Preliminary schedule -- subject to frequent adjustment.)

	Date	JDJ Reading	Topic	Assign.
1	01-16(Wed)	Chap. 1	Introduction, units and Poisson equation.	<a href="#">#1</a>
2	01-18(Fri)	Chap. 1	Electrostatic energy calculations	<a href="#">#2</a>
	01-21(Mon)	<i>No class</i>	<i>MKL Holiday</i>	
3	01-23(Wed)	Chap. 1	Poisson Equation and Green's Functions	<a href="#">#3</a>
4	01-25(Fri)	Chap. 1 & 2	Green's Theorem and Functions	<a href="#">#4</a>
5	01-28(Mon)	Chap. 1 & 2	Brief introduction to numerical methods	<a href="#">#5</a>
6	01-30(Wed)	Chap. 2	Method of images	<a href="#">#6</a>
7	02-01(Fri)	Chap. 3	Cylindrical and spherical geometries	<a href="#">#7</a>
8	02-04(Mon)	Chap. 4	Multipole moments	<a href="#">#8</a>
9	02-06(Wed)	Chap. 4	Dipoles and dielectrics	<a href="#">#9</a>
10	02-08(Fri)	Chap. 4	Microscopic and macroscopic polarizability	
11	02-11(Mon)	Chap. 5	Magnetostatics	<a href="#">#10</a>
12	02-13(Wed)	Chap. 5	Magnetostatic fields	
13	02-15(Fri)	Chap. 5	Magnetic dipole fields	Exam
14	02-18(Mon)	Chap. 5	Permeable media	Exam
15	02-20(Wed)	Chap. 6	Maxwell's equations	Exam
16	02-22(Fri)	Chap. 6	Poynting vector and stress tensor	Exam



Magnetic dipole moment

$$\mathbf{m} = \frac{1}{2} \int d^3r \mathbf{r} \times \mathbf{J}(\mathbf{r})$$

Note that the intrinsic spin of elementary particles is associated with a magnetic dipole moment, but we often do not have a detailed knowledge of  $\mathbf{J}(\mathbf{r})$ .

Vector potential for magnetic dipole moment

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{|\mathbf{r}|^3}$$

Macroscopic magnetization

$$\mathbf{M}(\mathbf{r}) = \sum_i \mathbf{m}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Vector potential due to “free” current  $\mathbf{J}_{\text{free}}(\mathbf{r})$  and macroscopic magnetization  $\mathbf{M}(\mathbf{r})$ . Note: the designation  $\mathbf{J}_{\text{free}}(\mathbf{r})$  implies that this current does not also contribute to the magnetization density.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \left( \frac{\mathbf{J}_{\text{free}}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

## Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \left( \frac{\mathbf{J}_{free}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right)$$

Note that :

$$\begin{aligned} \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} &= \mathbf{M}(\mathbf{r}') \times \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\ &= -\nabla' \times \left( \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) + \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ \Rightarrow \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \end{aligned}$$

## Vector potential contributions from macroscopic magnetization -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Note that for the case that  $\nabla \cdot \mathbf{A} = 0$ :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times (\nabla \times \mathbf{A}(\mathbf{r})) = -\nabla^2 \mathbf{A}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi} \int d^3 r' (4\pi \delta^3(\mathbf{r} - \mathbf{r}')) (\mathbf{J}_{free}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}'))$$

$$= \mu_0 (\mathbf{J}_{free}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r}))$$

$$\Rightarrow \nabla \times (\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

## Magnetic field contributions

$$\nabla \times (\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Define the magnetic flux density :

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

$$\nabla \times (\mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})) = \mu_0 \mathbf{J}_{free}(\mathbf{r})$$

Note that  $\mathbf{B}(\mathbf{r}) \equiv$  the magnetic flux density

Define  $\mathbf{H}(\mathbf{r}) \equiv$  the magnetic field

$$\mu_0 \mathbf{H}(\mathbf{r}) \equiv \mathbf{B}(\mathbf{r}) - \mu_0 \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

Summary of equations of magnetostatics :

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}_{total}(\mathbf{r})$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}_{free}(\mathbf{r})$$

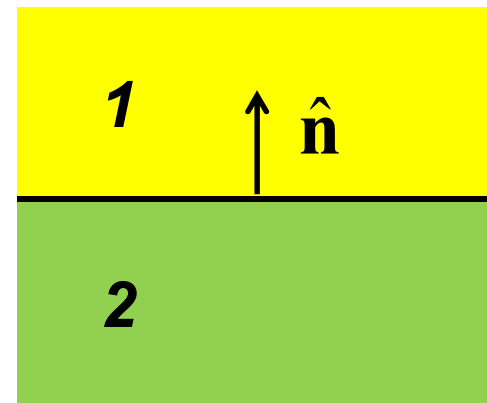
$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

For the case that  $\mathbf{J}_{free}(\mathbf{r})$ :

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$





For the case that  $\mathbf{J}_{free}(\mathbf{r})$ :

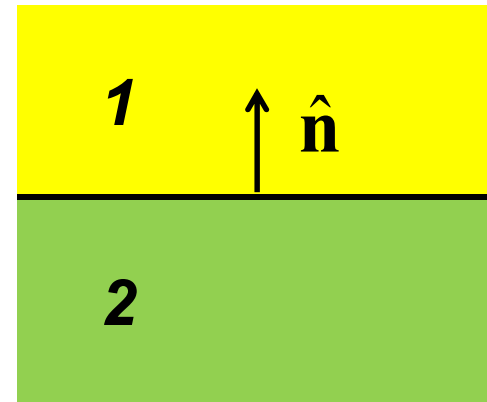
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$

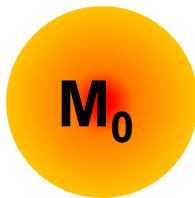
At boundary:

$$\mathbf{H}_1 \times \hat{\mathbf{n}} = \mathbf{H}_2 \times \hat{\mathbf{n}}$$

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} = \mathbf{B}_2 \cdot \hat{\mathbf{n}}$$



## Example magnetostatic boundary value problem



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

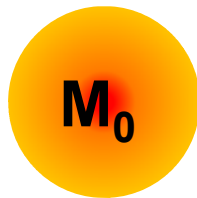
$$\nabla \times \mathbf{H}(\mathbf{r}) = 0 \quad \Rightarrow \quad \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 = \mu_0 \nabla \cdot (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\Rightarrow \nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

## Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases}$$

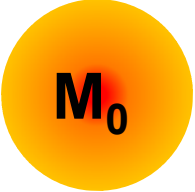
$$\nabla^2 \Phi_H(\mathbf{r}) = \nabla \cdot \mathbf{M}(\mathbf{r})$$

$$\Rightarrow \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \int d^3 r' \frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= -\frac{1}{4\pi} \int d^3 r' \left[ \nabla' \cdot \left( \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) - \mathbf{M}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right]$$

$$= -\frac{1}{4\pi} \nabla \cdot \int d^3 r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Example magnetostatic boundary value problem -- continued

  $\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ 0 & r > a \end{cases} \quad \Phi_H(\mathbf{r}) = -\frac{1}{4\pi} \nabla \cdot \int d^3 r' \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$

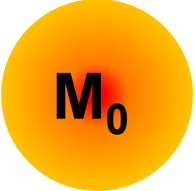
For this example:

$$\Phi_H(\mathbf{r}) = -\frac{M_0}{4\pi} \frac{\partial}{\partial z} \left( 4\pi \int_0^\infty r'^2 dr' \frac{1}{r_{>}} \right)$$

$$\text{For } r \leq a: \quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z} \left( \frac{a^2}{2} - \frac{r^2}{6} \right) = \frac{M_0 z}{3}$$

$$\text{For } r > a: \quad \Phi_H(\mathbf{r}) = -M_0 \frac{\partial}{\partial z} \left( \frac{a^3}{3r} \right) = \frac{M_0 a^3 z}{3r^3}$$

## Example magnetostatic boundary value problem -- continued



$$\mathbf{M}(\mathbf{r}) = \begin{cases} M_0 \hat{\mathbf{z}} & r \leq a \\ \mathbf{0} & r > a \end{cases}$$

$$\text{For } r \leq a: \quad \Phi_H(\mathbf{r}) = \frac{M_0 z}{3} \quad \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r}) = -\frac{M_0}{3} \hat{\mathbf{z}}$$

$$\text{For } r > a: \quad \Phi_H(\mathbf{r}) = \frac{M_0 a^3 z}{3r^3} \quad \mathbf{H}(\mathbf{r}) = -\nabla \Phi_H(\mathbf{r}) = -\frac{M_0 a^3}{3} \left( \frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 (\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r}))$$

$$\text{For } r \leq a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 \hat{\mathbf{z}}}{3} \quad \mathbf{B}(\mathbf{r}) = \mu_0 \frac{2M_0 \hat{\mathbf{z}}}{3}$$

$$\text{For } r > a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3} \left( \frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{B}(\mathbf{r}) = -\mu_0 \frac{M_0 a^3}{3} \left( \frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

Check boundary values:

$$\text{For } r \leq a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 \hat{\mathbf{z}}}{3} \quad \mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0}{3} \hat{\mathbf{z}} \times \hat{\mathbf{r}}$$

$$\text{For } r > a: \quad \mathbf{H}(\mathbf{r}) = -\frac{M_0 a^3}{3} \left( \frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

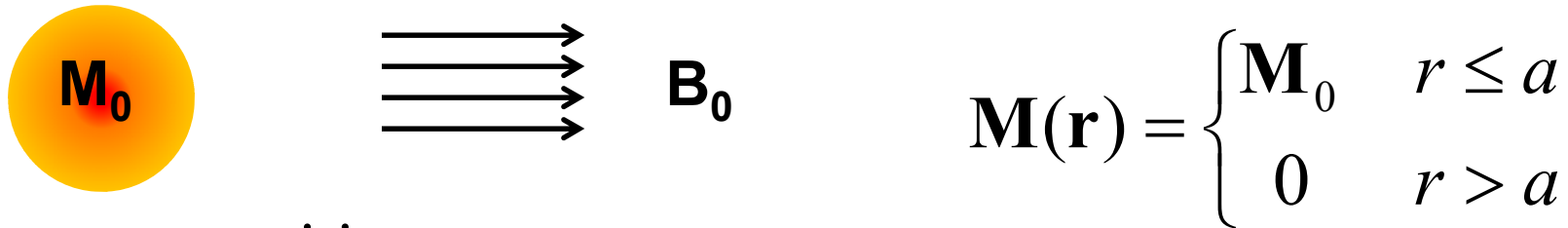
$$\mathbf{H}(a\hat{\mathbf{r}}) \times \hat{\mathbf{r}} = -\frac{M_0 a^3}{3} \frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{a^3}$$

$$\text{For } r \leq a: \quad \mathbf{B}(\mathbf{r}) = \mu_0 \frac{2M_0 \hat{\mathbf{z}}}{3} \quad \mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = \mu_0 \frac{2M_0}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}}$$

$$\text{For } r > a: \quad \mathbf{B}(\mathbf{r}) = -\mu_0 \frac{M_0 a^3}{3} \left( \frac{\hat{\mathbf{z}}}{r^3} - \frac{3z\mathbf{r}}{r^5} \right)$$

$$\mathbf{B}(a\hat{\mathbf{r}}) \cdot \hat{\mathbf{r}} = -\mu_0 \frac{M_0 a^3}{3} \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} \left( \frac{1}{a^3} - \frac{3a^2}{a^5} \right)$$

Variation; magnetic sphere plus external field  $\mathbf{B}_0$



By superposition :

For  $r \leq a$  :

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + \mu_0 \frac{2}{3} \mathbf{M}_0$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} \mathbf{B}_0 - \frac{1}{3} \mathbf{M}_0$$

$$\mathbf{B}(\mathbf{r}) + 2\mu_0 \mathbf{H}(\mathbf{r}) = 3\mathbf{B}_0$$

For an isotropic "paramagnetic" material,  $\mathbf{B}(\mathbf{r}) = \mu \mathbf{H}(\mathbf{r})$

$$\mathbf{M}_0 = \frac{3}{\mu_0} \left( \frac{\mu - \mu_0}{\mu + 2\mu_0} \right) \mathbf{B}_0$$