Electrodynamics – PHY712

Lecture 13 – Magnetic Dipolar Fields

Reference: Chap. 5 in J. D. Jackson's textbook.

Magnetic dipolar field

The magnetic dipole moment is defined by

$$\mathbf{m} = \frac{1}{2}\int d^3r'\,\mathbf{r}' \times \mathbf{J}(\mathbf{r}'), \tag{1}$$

with the corresponding potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}, \tag{2}$$

and magnetostatic field

$$\mathbf{B}_{\mathbf{m}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\frac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}})-\mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\delta^3(\mathbf{r})\right\}. \tag{3}$$

Magnetic dipolar field – continued

Some details:

$$\nabla \times (s\mathbf{V}) = \nabla s \times \mathbf{V} + s\nabla \times \mathbf{V}. \tag{4}$$

$$\nabla \times (\mathbf{V}_1 \times \mathbf{V}_2) = \mathbf{V}_1(\nabla \cdot \mathbf{V}_2) - \mathbf{V}_2(\nabla \cdot \mathbf{V}_1) + (\mathbf{V}_2 \cdot \nabla)\mathbf{V}_1 - (\mathbf{V}_1 \cdot \nabla)\mathbf{V}_2. \tag{5}$$

For $r > 0$:

$$\nabla \times \left(\frac{\mathbf{m} \times \mathbf{r}}{r^3}\right) = \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r}) - r^2\mathbf{m}}{r^5}. \tag{6}$$

Justification for the δ function contribution at the origin of the magnetic dipole

Note: This derivation is very similar to the analogous electrostatic case.

The evaluation of the field at the origin of the dipole is poorly defined, but we make the following approximation.

$$\mathbf{B}(\mathbf{r} \approx \mathbf{0}) \approx \left(\int_{\text{sphere}} \mathbf{B}(\mathbf{r}) \mathbf{d^3 r} \right) \delta^3(\mathbf{r}). \tag{7}$$

First we note that

$$\int_{r \leq R} \mathbf{B}(\mathbf{r}) d^3 r = R^2 \int_{r=R} \hat{\mathbf{r}} \times \mathbf{A}(\mathbf{r}) \, d\Omega. \tag{8}$$

This result follows from the divergence theorm:

$$\int_{\text{vol}} \nabla \cdot \mathcal{V} \mathbf{d^3 r} = \int_{\text{surface}} \mathcal{V} \cdot \mathbf{dA}. \tag{9}$$

Singular contribution to dipolar field – continued

The divergence theorem can be used to prove Eq. (8) for each cartesian coordinate of $\nabla \times \mathbf{A}$ since $\nabla \times \mathbf{A} = \hat{\mathbf{x}}\,(\hat{\mathbf{x}} \cdot (\nabla \times \mathbf{A})) + \hat{\mathbf{y}}\,(\hat{\mathbf{y}} \cdot (\nabla \times \mathbf{A})) + \hat{\mathbf{z}}\,(\hat{\mathbf{z}} \cdot (\nabla \times \mathbf{A}))$. Note that $\hat{\mathbf{x}} \cdot (\nabla \times \mathbf{A}) = -\nabla \cdot (\hat{\mathbf{x}} \times \mathbf{A})$ and that we can use the Divergence theorem with $\mathcal{V} \equiv \hat{\mathbf{x}} \times \mathbf{A}(\mathbf{r})$ for the $x-$ component for example:

$$\int_{\text{vol}} \nabla \cdot (\hat{\mathbf{x}} \times \mathbf{A}) d^3r = \int_{\text{surface}} (\hat{\mathbf{x}} \times \mathbf{A}) \cdot \hat{\mathbf{r}} dA = \int_{\text{surface}} (\mathbf{A} \times \hat{\mathbf{r}}) \cdot \hat{\mathbf{x}} dA. \tag{10}$$

Therefore,

$$\int_{r \leq R} (\nabla \times \mathbf{A}) d^3r = -\int_{r=R} (\mathbf{A} \times \hat{\mathbf{r}}) \cdot (\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}) dA = R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega \tag{11}$$

which is identical to Eq. (8). We can use the identity (as in electrostatic case),

$$\int d\Omega \frac{\hat{\mathbf{r}}}{|\mathbf{r} - \mathbf{r}'|} = \frac{4\pi}{3} \frac{r_<}{r_>^2}\, \hat{\mathbf{r}}'. \tag{12}$$

Singular contribution to dipolar field – continued

Now, expressing the vector potential in terms of the current density:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{13}$$

the integral over Ω in Eq. 8 becomes

$$R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega = \frac{4\pi R^2}{3}\frac{\mu_0}{4\pi}\int d^3r' \, \frac{r_<}{r_>^2}\, \hat{\mathbf{r}'} \times \mathbf{J}(\mathbf{r}'). \tag{14}$$

If the sphere R contains the entire current distribution, then $r_> = R$ and $r_< = r'$ so that (14) becomes

$$R^2 \int_{r=R} (\hat{\mathbf{r}} \times \mathbf{A}) d\Omega = \frac{4\pi}{3}\frac{\mu_0}{4\pi}\int d^3r' \, \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \equiv \frac{8\pi}{3}\frac{\mu_0}{4\pi}\mathbf{m}, \tag{15}$$

which thus justifies the delta-function contribution in Eq. 3 and results so-called “Fermi contact” contribution in the “hyperfine” interaction.

Magnetic field due to electrons in the vicinity of a nucleus

Contribution due to "orbital" magnetism in a spherical atom

The current density associated with an electron in a bound state of an atom as described by a quantum mechanical wavefunction $\psi_{nlm_l}(\mathbf{r})$ can be written:

$$\mathbf{J}(\mathbf{r}) = \frac{-e\hbar m_l \hat{\phi}}{m_e r \sin\theta} \left|\psi_{nlm_l}(\mathbf{r})\right|^2 . \tag{16}$$

In the following, it will be convenient to represent the azimuthal unit vector $\hat{\phi}$ in terms of cartesian coordinates:

$$\hat{\phi} = -\sin\phi \hat{\mathbf{x}} + \cos\phi \hat{\mathbf{y}} = \frac{\hat{\mathbf{z}} \times \mathbf{r}}{r \sin\theta}. \tag{17}$$

The vector potential for this current density can be written

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e} m_l \int d^3r' \frac{\hat{\mathbf{z}} \times \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \frac{\left|\psi_{nlm_l}(\mathbf{r}')\right|^2}{r'^2 \sin^2\theta'} \tag{18}$$

Contribution due to "orbital" magnetism in a spherical atom – continued

We want to evaluate the magnetic field $B = \nabla \times A$ in the vicinity of the nucleus $(\mathbf{r} \rightarrow 0)$. Taking the curl of the Eq. 18, we obtain

$$\mathbf{B_o}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\hbar}{m_e}\, m_l \int d^3r' \frac{(\mathbf{r}-\mathbf{r}') \times (\hat{\mathbf{z}} \times \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{\mathbf{3}}} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'} \tag{19}$$

Evaluating this expression with $(\mathbf{r} \rightarrow 0)$, we obtain

$$\mathbf{B_o}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e}\, m_l \int d^3r' \frac{\mathbf{r}' \times (\hat{\mathbf{z}} \times \mathbf{r}')}{r'^3} \frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2 \sin^2\theta'} \tag{20}$$

Contribution due to "orbital" magnetism in a spherical atom – continued

$$\mathbf{B_o}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e}\,m_l\int d^3r'\,\frac{\mathbf{r}'\times(\hat{\mathbf{z}}\times\mathbf{r}')}{r'^3}\,\frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2\sin^2\theta'} \tag{21}$$

Expanding the cross product and expressing the result in spherical polar coordinates, we obtain in the numerator

$\hat{\mathbf{r}}'\times(\hat{\mathbf{z}}\times\hat{\mathbf{r}}') = \hat{\mathbf{z}}(\mathbf{1}-\cos^{\mathbf{2}}\theta') - \hat{\mathbf{x}}\cos\theta'\sin\theta'\cos\phi' - \hat{\mathbf{y}}\cos\theta'\sin\theta'\sin\phi')$.

In evaluating the integration over the azimuthal variable ϕ', the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components vanish which reduces to

$$\mathbf{B_o}(\mathbf{0}) = -\frac{\mu_0}{4\pi}\frac{e\hbar}{m_e}\,m_l\int d^3r'\,\frac{\hat{\mathbf{z}}r'^2\sin^2\theta'}{r'^3}\,\frac{|\psi_{nlm_l}(\mathbf{r}')|^2}{r'^2\sin^2\theta'} \tag{22}$$

and

$$\mathbf{B_o}(\mathbf{0}) = -\frac{\mu_0 e\hbar m_l\hat{\mathbf{z}}}{4\pi m_e}\int d^3r'\,|\psi_{nlm_l}|^2\,\frac{1}{r'^3} \equiv -\frac{\mu_0 e}{4\pi m_e}L_z\hat{\mathbf{z}}\left\langle\frac{1}{r'^3}\right\rangle. \tag{23}$$

“Hyperfine” interaction

The so-called “hyperfine” interaction results from the magnetic dipole moment of a nucleus $\mu_{\mathbf{N}}$ responding to the magnetic field formed by the magnetic dipole of the electron spin ($\mu_{\mathbf{e}}$) as well as the electron orbital current contribution.

$$\mathcal{H}_{\mathrm{HF}} = -\mu_{\mathbf{N}} \cdot \left(\mathbf{B}_{\mu_e} + \mathbf{B}_o(0)\right). \tag{24}$$

$$\mathcal{H}_{\mathrm{HF}} = -\frac{\mu_0}{4\pi}\left(\frac{3(\mu_{\mathbf{N}}\cdot\hat{\mathbf{r}})(\mu_{\mathbf{e}}\cdot\hat{\mathbf{r}}) - \mu_{\mathbf{N}}\cdot\mu_{\mathbf{e}}}{r^3} + \frac{8\pi}{3}\mu_{\mathbf{N}}\cdot\mu_{\mathbf{e}}\delta^3(\mathbf{r}) + \frac{e}{m_e}\left\langle\frac{\mathbf{L}\cdot\mu_{\mathbf{N}}}{r^3}\right\rangle\right). \tag{25}$$