PHY 712 Electrodynamics
11-11:50 AM MWF Olin 107

Plan for Lecture 11:

Start reading Chapter 5

A. Magnetostatics

B. Vector potential

C. Example: current loop

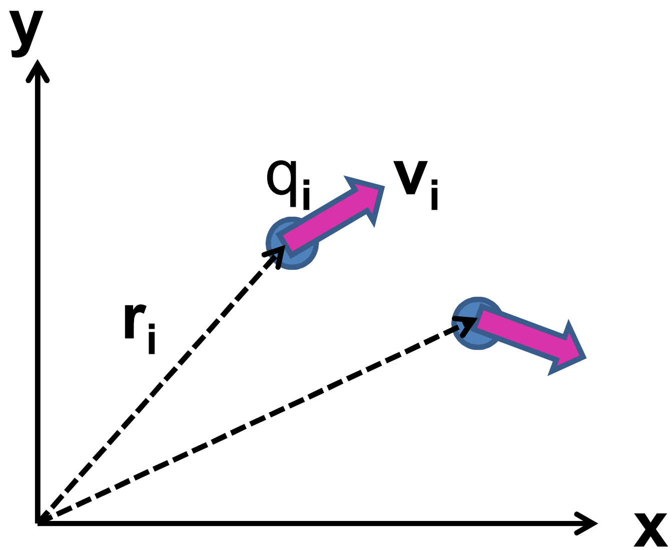
Course schedule for 2013

(Preliminary schedule -- subject to frequent adjustment.)

	Date	JDJ Reading	Topic	Assign.
1	01-16(Wed)	Chap. 1	Introduction, units and Poisson equation.	#1
2	01-18(Fri)	Chap. 1	Electronstatic energy calculations	#2
	01-21(Mon)	*No class*	*MKL Holiday*	
3	01-23(Wed)	Chap. 1	Poisson Equation and Green's Functions	#3
4	01-25(Fri)	Chap. 1 & 2	Green's Theorem and Functions	#4
5	01-28(Mon)	Chap. 1 & 2	Brief introduction to numerical methods	#5
6	01-30(Wed)	Chap. 2	Method of images	#6
7	02-01(Fri)	Chap. 3	Cylindrical and spherical geometries	#7
8	02-04(Mon)	Chap. 4	Multipole moments	#8
9	02-06(Wed)	Chap. 4	Dipoles and dielectrics	#9
10	02-08(Fri)	Chap. 4	Microscopic and macroscopic polarizability	
11	02-11(Mon)	Chap. 4	Magnetostatics	#10

Magnetostatics

Magnetic flux density or magnetic induction field **B**
Steady state (time constant) current density **J**



$$\mathbf{J}(\mathbf{r}) = \sum_i q_i \mathbf{v}_i \delta^3(\mathbf{r} - \mathbf{r}_i)$$

Note that "statics" implies that $\nabla \cdot \mathbf{J} = 0$.

This follows from the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Comparison of electrostatics and magnetostatics

Electrostatic field due to charge density $\rho(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

Alternative forms magnetostatic equations

Magnetostatic field due to current density $\mathbf{J}(\mathbf{r})$:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

Note that: $\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$

Also note that: $\nabla\times\left(s(\mathbf{r})\mathbf{V}(\mathbf{r})\right) = \nabla s(\mathbf{r})\times\mathbf{V}(\mathbf{r}) + s(\mathbf{r})\nabla\times\mathbf{V}(\mathbf{r})$

$$\Rightarrow \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla\times\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Alternative forms magnetostatic equations -- continued

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\Rightarrow \nabla \cdot \mathbf{B}(\mathbf{r}) = 0$$ No magnetic monopoles

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$ Ampere's law

"Proof" of Ampere's law for magnetostatic system:

$$\nabla \times \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \nabla \times \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Note that: $\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$

Recall that: $\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')$ and $\nabla \cdot \mathbf{J}(\mathbf{r}) = 0$

Differential forms of magnetostatic equations:

$$\Rightarrow \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \text{No magnetic monopoles}$$

$$\Rightarrow \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}) \qquad \text{Ampere's law}$$

Magnetostatic vector potential

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\Rightarrow \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \nabla s(\mathbf{r})$$

Non uniqueness of the magnetostatic vector potential

$$\text{Note that:} \quad \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \nabla \times \mathbf{A}'(\mathbf{r})$$

$$\text{if} \quad \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla s(\mathbf{r})$$

$$\text{Example:} \quad \text{for } \mathbf{B}(\mathbf{r}) = B_0 \hat{\mathbf{z}}$$

$$\mathbf{A}(\mathbf{r}) = \tfrac{1}{2} B_0 (x\hat{\mathbf{y}} - y\hat{\mathbf{x}})$$

$$\text{or} \quad \mathbf{A}(\mathbf{r}) = B_0 x \hat{\mathbf{y}}$$

$$\text{or} \quad \mathbf{A}(\mathbf{r}) = -B_0 y \hat{\mathbf{x}}$$

Differential form of Ampere's law in terms of vector potential:

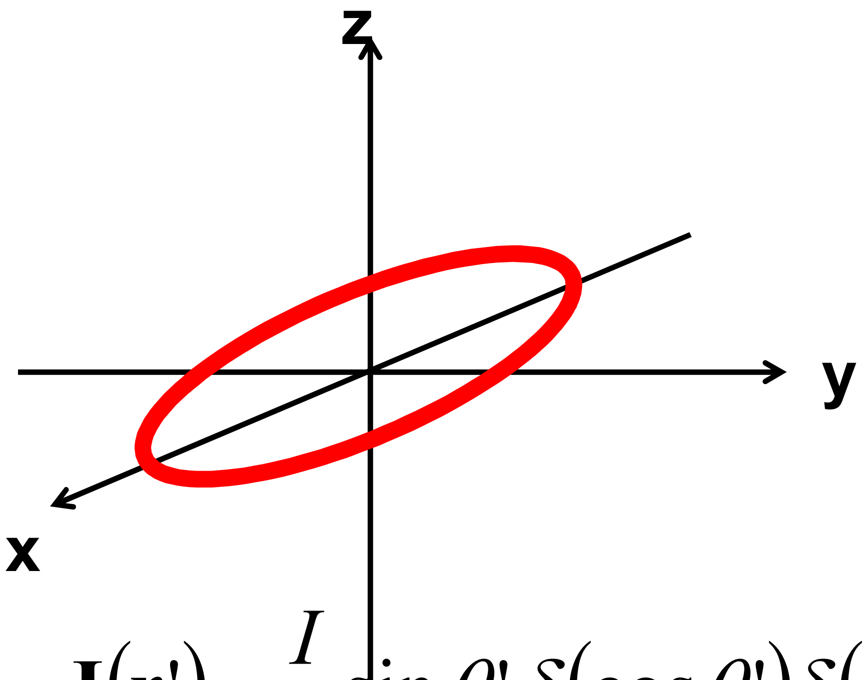
$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}(\mathbf{r})) - \nabla^2 \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r})$$

$$\text{If} \quad \nabla \cdot \mathbf{A}(\mathbf{r}) = 0 \quad \text{(Coulomb gauge)} \quad \Rightarrow \quad \nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

Magnetostatics example: current loop



$$\mathbf{J}(\mathbf{r}') = \frac{I}{a}\sin\theta'\,\delta(\cos\theta')\,\delta(r'-a)\left(-\sin\phi'\,\hat{\mathbf{x}} + \cos\phi'\,\hat{\mathbf{y}}\right)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

Magnetostatics example: current loop -- continued

$$\mathbf{J}(\mathbf{r}') = \frac{I}{a}\sin\theta'\,\delta(\cos\theta')\,\delta(r'-a)\left(-\sin\phi'\,\hat{\mathbf{x}} + \cos\phi'\,\hat{\mathbf{y}}\right)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi a}\int r'^2\,dr'\,d\cos\theta'\,d\phi'\,\frac{\sin\theta'\,\delta(\cos\theta')\,\delta(r'-a)\left(-\sin\phi'\,\hat{\mathbf{x}} + \cos\phi'\,\hat{\mathbf{y}}\right)}{\left(r^2 + r'^2 - 2rr'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right)\right)^{1/2}}$$

Completing integration over r' and θ':

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I a^2}{4\pi a}\int_0^{2\pi} d\phi'\,\frac{\left(-\sin\phi'\,\hat{\mathbf{x}} + \cos\phi'\,\hat{\mathbf{y}}\right)}{\left(r^2 + a^2 - 2ra\left(\sin\theta\cos(\phi-\phi')\right)\right)^{1/2}}$$

Let $\phi - \phi' \equiv \varphi$

$$\sin\phi' = \sin(\phi-\varphi) = \sin\phi\cos\varphi - \cos\phi\sin\varphi$$

$$\cos\phi' = \cos(\phi-\varphi) = \cos\phi\cos\varphi + \sin\phi\sin\varphi$$

Remaining non - trivial terms

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I a}{4\pi}\left(\sin\phi\,\hat{\mathbf{x}} - \cos\phi\,\hat{\mathbf{y}}\right)\int_0^{2\pi} d\varphi\,\frac{\cos\varphi}{\left(r^2 + a^2 - 2ra\left(\sin\theta\cos\varphi\right)\right)^{1/2}}$$

Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I a}{4\pi}(\sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}})\int_0^{2\pi} d\varphi \frac{\cos\varphi}{\left(r^2 + a^2 - 2ra(\sin\theta\cos\varphi)\right)^{1/2}}$$

Elliptic integrals:

$$K(m) = \int_0^{\pi/2} \frac{du}{\left(1 - m\sin^2 u\right)^{1/2}}$$

$$E(m) = \int_0^{\pi/2} \left(1 - m\sin^2 u\right)^{1/2} du$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} 4Ia \frac{(\sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}})}{\left(r^2 + a^2 + 2ra\sin\theta\right)^{1/2}} \left[\frac{\left(2 - k^2\right)K(k) - 2E(k)}{k^2}\right]$$

where: $\quad k \equiv \dfrac{4ar\sin\theta}{r^2 + a^2 + 2ra\sin\theta}$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

Magnetostatics example: current loop -- continued

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} 4Ia \frac{(\sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}})}{(r^2 + a^2 + 2ra\sin\theta)^{1/2}} \left[\frac{(2-k^2)K(k) - 2E(k)}{k^2} \right]$$

$$\text{where:} \qquad k \equiv \frac{4ar\sin\theta}{r^2 + a^2 + 2ra\sin\theta}$$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$$

Note that for spherical polar coordinates: $\hat{\boldsymbol{\varphi}} = \sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}}$

$$\mathbf{A}(\mathbf{r}) = A_\phi(\mathbf{r})\hat{\boldsymbol{\varphi}}$$

$$\text{where} \quad A_\phi(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{4Ia}{(r^2 + a^2 + 2ra\sin\theta)^{1/2}} \left[\frac{(2-k^2)K(k) - 2E(k)}{k^2} \right]$$

$$\mathbf{B}(\mathbf{r}) = \frac{1}{r\sin\theta} \frac{\partial(\sin\theta A_\phi(\mathbf{r}))}{\partial\theta}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial(rA_\phi(\mathbf{r}))}{\partial r}\hat{\boldsymbol{\theta}}$$

For $r \to \infty$:

$$\mathbf{B}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)$$

Other examples of current density sources:

Quantum mechanical expression for current density

for a particle of mass M and charge e and of probability amplitude $\Psi(\mathbf{r})$:

$$\mathbf{J}(\mathbf{r}) = -\frac{e\hbar}{2Mi}\left(\Psi^*(\mathbf{r})\nabla\Psi(\mathbf{r}) - \Psi(\mathbf{r})\nabla\Psi^*(\mathbf{r})\right)$$

For an electron in a spherical potential (such as in an atom):

$$\Psi(\mathbf{r}) \equiv \Psi_{nlm}(\mathbf{r}) = R_{nl}(r)Y_{lm}(\hat{\mathbf{r}})$$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{2Mi}\left|R_{nl}(r)\right|^2 \frac{1}{r\sin\theta}\left(Y_{lm}{}^*(\hat{\mathbf{r}})\frac{\partial Y_{lm}(\hat{\mathbf{r}})}{\partial\phi} - Y_{lm}(\hat{\mathbf{r}})\frac{\partial Y_{lm}{}^*(\hat{\mathbf{r}})}{\partial\phi}\right)\hat{\boldsymbol{\varphi}}$$

$$= \frac{e\hbar}{M}\frac{m}{r\sin\theta}\left|\Psi_{nlm}(\mathbf{r})^2\right|\hat{\boldsymbol{\varphi}}$$

Note that: $\hat{\boldsymbol{\varphi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} = \dfrac{\hat{\mathbf{z}}\times\mathbf{r}}{r\sin\theta}$

$$\mathbf{J}(\mathbf{r}) = \frac{e\hbar}{M}\frac{m}{r^2\sin^2\theta}\left|\Psi_{nlm}(\mathbf{r})^2\right|(\hat{\mathbf{z}}\times\mathbf{r})$$