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Additional Notes for Lecture #6 – Mean value theorem for solutions to the Laplace equation

Consider an electrostatic field $\Phi(\mathbf{r})$ in a charge-free region so that it satisfies the Laplace equation:

$$\nabla^2\Phi(\mathbf{r}) = 0. \tag{1}$$

The "mean value theorem" value theorem (problem 1.10 of your textbook) states that the value of $\Phi(\mathbf{r})$ at the arbitrary (charge-free) point $\mathbf{r}$ is equal to the average of $\Phi(\mathbf{r}')$ over the surface of any sphere centered on the point $\mathbf{r}$ (see Jackson problem #1.10). One way to prove this theorem is the following. Consider a point $\mathbf{r}' = \mathbf{r} + \mathbf{u}$, where $\mathbf{u}$ will describe a sphere of radius R about the fixed point $\mathbf{r}$. We can make a Taylor series expansion of the electrostatic potential $\Phi(\mathbf{r}')$ about the fixed point $\mathbf{r}$:

$$\Phi(\mathbf{r}+\mathbf{u}) = \Phi(\mathbf{r}) + \mathbf{u}\cdot\nabla\Phi(\mathbf{r}) + \frac{1}{2!}(\mathbf{u}\cdot\nabla)^2\Phi(\mathbf{r}) + \frac{1}{3!}(\mathbf{u}\cdot\nabla)^3\Phi(\mathbf{r}) + \frac{1}{4!}(\mathbf{u}\cdot\nabla)^4\Phi(\mathbf{r}) + \cdots. \tag{2}$$

According to the premise of the theorem, we want to integrate both sides of the equation 2 over a sphere of radius R in the variable $\mathbf{u}$:

$$\int_{\text{sphere}} dS_u = R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u). \tag{3}$$

We note that

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u) 1 = 4\pi R^2, \tag{4}$$

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)\mathbf{u}\cdot\nabla = 0, \tag{5}$$

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)(\mathbf{u}\cdot\nabla)^2 = \frac{4\pi R^4}{3}\nabla^2, \tag{6}$$

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)(\mathbf{u}\cdot\nabla)^3 = 0, \tag{7}$$

and

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)(\mathbf{u}\cdot\nabla)^4 = \frac{4\pi R^6}{5}\nabla^4. \tag{8}$$

Since $\nabla^2\Phi(\mathbf{r}) = 0$, the only non-zero term of the average it thus the first term:

$$R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)\Phi(\mathbf{r}+\mathbf{u}) = 4\pi R^2\Phi(\mathbf{r}), \tag{9}$$

or

$$\Phi(\mathbf{r}) = \frac{1}{4\pi R^2}\; R^2\int_0^{2\pi} d\phi_u \int_{-1}^{+1} d\cos(\theta_u)\Phi(\mathbf{r}+\mathbf{u}) \equiv \frac{1}{4\pi R^2}\int_{\text{sphere}} dS_u\Phi(\mathbf{r}+\mathbf{u}). \tag{10}$$

Since this result is independent of the radius R, we see that we have proven the theorem.