

Notes for Lecture #21

Derivation of the Lienard-Wiechert potentials and fields

Consider a point charge q moving on a trajectory $\mathbf{R}_q(t)$. We can write its charge density as

$$\rho(\mathbf{r}, t) = q\delta^3(\mathbf{r} - \mathbf{R}_q(t)), \quad (1)$$

and the current density as

$$\mathbf{J}(\mathbf{r}, t) = q\dot{\mathbf{R}}_q(t)\delta^3(\mathbf{r} - \mathbf{R}_q(t)), \quad (2)$$

where

$$\dot{\mathbf{R}}_q(t) \equiv \frac{d\mathbf{R}_q(t)}{dt}. \quad (3)$$

Evaluating the scalar and vector potentials in the Lorentz gauge,

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \int d^3r' dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c)), \quad (4)$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int \int d^3r' dt' \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} \delta(t' - (t - |\mathbf{r} - \mathbf{r}'|/c)). \quad (5)$$

We perform the integrations over first d^3r' and then dt' , and make use of the fact that for any function of t' ,

$$\int_{-\infty}^{\infty} dt' f(t') \delta(t' - (t - |\mathbf{r} - \mathbf{R}_q(t')|/c)) = \frac{f(t_r)}{1 - \frac{\dot{\mathbf{R}}_q(t_r) \cdot (\mathbf{r} - \mathbf{R}_q(t_r))}{c|\mathbf{r} - \mathbf{R}_q(t_r)|}}, \quad (6)$$

where the “retarded time” is defined to be

$$t_r \equiv t - \frac{|\mathbf{r} - \mathbf{R}_q(t_r)|}{c}. \quad (7)$$

We find

$$\Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}, \quad (8)$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\mathbf{v}}{R - \frac{\mathbf{v} \cdot \mathbf{R}}{c}}, \quad (9)$$

where we have used the shorthand notation $\mathbf{R} \equiv \mathbf{r} - \mathbf{R}_q(t_r)$ and $\mathbf{v} \equiv \dot{\mathbf{R}}_q(t_r)$.

In order to find the electric and magnetic fields, we need to evaluate

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t} \quad (10)$$

and

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t). \quad (11)$$

The trick of evaluating these derivatives is that the retarded time (7) depends on position \mathbf{r} and on itself. We can show the following results using the shorthand notation defined above:

$$\nabla t_r = -\frac{\mathbf{R}}{c \left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)}, \quad (12)$$

and

$$\frac{\partial t_r}{\partial t} = \frac{R}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)}. \quad (13)$$

Evaluating the gradient of the scalar potential, we find:

$$-\nabla \Phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^3} \left[\mathbf{R} \left(1 - \frac{v^2}{c^2} \right) - \frac{\mathbf{v}}{c} \left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right) + \mathbf{R} \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right], \quad (14)$$

and

$$-\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^3} \left[\frac{\mathbf{v}R}{c} \left(\frac{v^2}{c^2} - \frac{\mathbf{v} \cdot \mathbf{R}}{Rc} - \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right) - \frac{\dot{\mathbf{v}}R}{c^2} \left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right) \right]. \quad (15)$$

These results can be combined to determine the electric field:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^3} \left[\left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \left(1 - \frac{v^2}{c^2} \right) + \left(\mathbf{R} \times \left\{ \left(\mathbf{R} - \frac{\mathbf{v}R}{c} \right) \times \frac{\dot{\mathbf{v}}}{c^2} \right\} \right) \right]. \quad (16)$$

We can also evaluate the curl of \mathbf{A} to find the magnetic field:

$$\mathbf{B}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \left[\frac{-\mathbf{R} \times \mathbf{v}}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^3} \left(1 - \frac{v^2}{c^2} + \frac{\dot{\mathbf{v}} \cdot \mathbf{R}}{c^2} \right) - \frac{\mathbf{R} \times \dot{\mathbf{v}}/c}{\left(R - \frac{\mathbf{v} \cdot \mathbf{R}}{c} \right)^2} \right]. \quad (17)$$

One can show that the electric and magnetic fields are related according to

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{R} \times \mathbf{E}(\mathbf{r}, t)}{cR}. \quad (18)$$