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PHY 711 – Contour Integration

These notes summarize some basic properties of complex functions and their integrals. An *analytic* function $f(z)$ in a certain region of the complex plane z is one which takes a single (non-infinite) value and is differentiable within that region. Cauchy's theorm states that a closed contour integral of the function within that region has the value

$$\oint_C f(z) = 0. \tag{1}$$

As an example, functions composed of integer powers of z –

$$f(z) = z^n, \quad \text{for} \quad n = 0, 1, \pm 2, \pm 3.... \tag{2}$$

fall in this catogory. Notice that non-integral powers are generally not analytic and that $n = -1$ is also special. In fact, we can show that

$$\oint_C \frac{dz}{z} = 2\pi i. \tag{3}$$

This result follows from the fact that we can deform the contour to a unit circle about the origin so that $z = \mathrm{e}^{i\theta}$. Then

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{\mathrm{e}^{i\theta}}{\mathrm{e}^{i\theta}} i d\theta = 2\pi i. \tag{4}$$

One result of this analysis is the Cauchy integral formula which states that for any analytic function $f(z)$ within a region C,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz'. \tag{5}$$

Another result of this analysis is the Residue Theorm which states that if the complex function $g(z)$ has poles at a finite number of points z_p within a region C but is otherwise analytic, the contour integral can be avaluated according to

$$\oint_C g(z) dz = 2\pi i \sum_p Res(g_p), \tag{6}$$

where the residue is given by

$$Res(g_p) \equiv \lim_{z \to z_p} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_p)^m g(z) \right) \right\}, \tag{7}$$

where m denotes the order of the pole.