PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 7:

Continue reading Chapter 3

1. **Lagrange's equations**
2. **D'Alembert's principle**

Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/28/2013	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/30/2013	Chap. 1	Scattering theory continued	#2
3	Mon, 9/02/2013	Chap. 1	Scattering theory continued	#3
4	Wed, 9/04/2013	Chap. 2	Accelerated Coordinate Systems	#4
5	Fri, 9/06/2013	Chap. 3	Calculus of variations	#5
6	Mon, 9/09/2013	Chap. 3	Calculus of variations -- continued	
7	Wed, 9/11/2013	Chap. 3	Calculus of variations applied to Lagrangians	#6
8	Fri, 9/13/2013	Chap. 3	Lagrangian mechanics	#7

FOREST
RSITY

Department of Physics

WFU Physics Colloquium

TITLE: "WFU Physics Research -- Part II"
TIME: Wednesday Sept. 11, 2013 at 4:00 PM
PLACE: George P. Williams, Jr. Lecture Hall, (Olin 101)

Refreshments will be served at 3:30 PM in the lounge. All interested persons are cordially invited to attend.

PROGRAM

This colloquium is the second of two which will highlight physics research at Wake Forest University. During the colloquium, Physics Department faculty members will present short overviews of their research programs in the Physics Department. This forum for sharing ideas will hopefully inspire collaborations between students and faculty and between research groups.

Summary of results from the calculus of variation

For the class of problems where we need to perform an extremization on an integral form:

$$I = \int_{x_i}^{x_f} f\left(\left\{y(x), \frac{dy}{dx}\right\}, x\right) dx$$

A necessary condition is the Euler - Lagrange equations:

$$\left(\frac{\partial f}{\partial y}\right) - \frac{d}{dx}\left[\left(\frac{\partial f}{\partial (dy/dx)}\right)\right] = 0$$

$$\frac{d}{dx}\left(f - \frac{\partial f}{\partial (dy/dx)}\frac{dy}{dx}\right) = \left(\frac{\partial f}{\partial x}\right)$$

Application to particle dynamics

Simple example: vertical trajectory of particle of mass *m* subject to constant downward acceleration *a=-g*.

$$m\frac{d^2 y}{dt^2} = -mg$$

$$y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Now consider the Lagrangian defined to be:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U$$

Kinetic energy (T)   Potential energy (U)

In our example:

$$L\left(\left\{y(t), \frac{dy}{dt}\right\}, t\right) \equiv T - U = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy$$

Hamilton's principle states:

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 - mgy\right) dt \quad \text{is minimized for physical } y(t):$$

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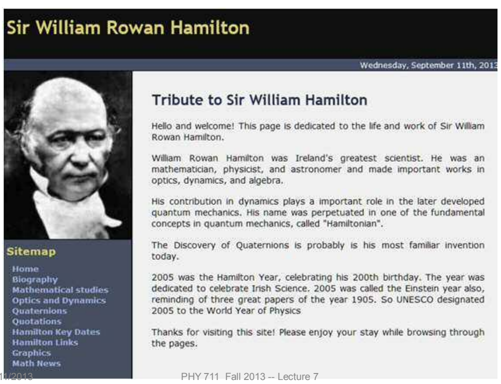
http://www.hamilton2005.ie/



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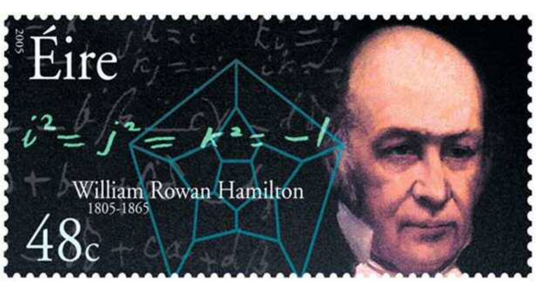

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Condition for minimizing the action :

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Euler - Lagrange relations :

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$\Rightarrow -mg - \frac{d}{dt} m\dot{y} = 0$$

$$\Rightarrow \frac{d}{dt}\frac{dy}{dt} = -g$$

$$y(t) = y_i + v_i t - \tfrac{1}{2} g t^2$$

Check :

$$S \equiv \int_{t_i}^{t_f} \left(\frac{1}{2} m \left(\frac{dy}{dt} \right)^2 - mgy \right) dt$$

Assume $t_i = 0,\ y_i = H \equiv \tfrac{1}{2} g T^2;\ \ t_f = T,\ y_f = 0$

Trial trajectories : $y_1(t) = \tfrac{1}{2} g T^2 (1 - t/T) = H - \tfrac{1}{2} g T t$

$y_2(t) = \tfrac{1}{2} g T^2 (1 - t^2/T^2) = H - \tfrac{1}{2} g t^2$

$y_3(t) = \tfrac{1}{2} g T^2 (1 - t^3/T^3) = H - \tfrac{1}{2} g t^3 / T$

Maple says :

$$S_1 = -0.125 m g^2 T^3$$
$$S_2 = -0.167 m g^2 T^3$$
$$S_3 = -0.150 m g^2 T^3$$

Jean d'Alembert 1717-1783

D'Alembert's principle:

Generalized coordinates : $q_\sigma(\{x_i\})$

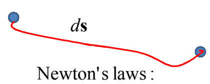
Newton's laws :

$\mathbf{F}\text{-}m\mathbf{a} = 0 \qquad \Rightarrow (\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s} = 0$

$$\mathbf{F}\cdot d\mathbf{s} = \sum_\sigma \sum_i F_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

For a conservative force : $F_i = -\frac{\partial U}{\partial x_i}$

$$\mathbf{F}\cdot d\mathbf{s} = -\sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$



Generalized coordinates : $q_\sigma(\{x_i\})$

Newton's laws :

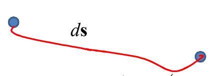
$\mathbf{F}\text{-}m\mathbf{a} = 0 \qquad \Rightarrow (\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s} = 0$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i m\ddot{x}_i \frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma$$

$$= \sum_\sigma \sum_i \left(\frac{d}{dt}\left(m\dot{x}_i \frac{\partial x_i}{\partial q_\sigma}\right) - m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma}\right)\delta q_\sigma$$

Claim : $\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial \dot{x}_i}{\partial \dot{q}_\sigma}$ and $\frac{d}{dt}\frac{\partial x_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\sigma}\frac{dx_i}{dt} \equiv \frac{\partial \dot{x}_i}{\partial q_\sigma}$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial \dot{q}_\sigma}\right) - \frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial q_\sigma}\right)\delta q_\sigma$$



Generalized coordinates : $q_\sigma(\{x_i\})$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \sum_i \left(\frac{d}{dt}\left(\frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial \dot{q}_\sigma}\right) - \frac{\partial\left(\frac{1}{2}m\dot{x}_i^2\right)}{\partial q_\sigma}\right)\delta q_\sigma$$

Define -- kinetic energy : $T \equiv \sum_i \frac{1}{2}m\dot{x}_i^2$

$$m\mathbf{a}\cdot d\mathbf{s} = \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}\right)\delta q_\sigma$$

Recall :

$$\mathbf{F}\cdot d\mathbf{s} = \sum_\sigma \sum_i \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial q_\sigma}\delta q_\sigma = \sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma$$

$$(\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s} = \sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

Generalized coordinates : $q_\sigma(\{x_i\})$

$$(\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s} = -\sum_\sigma \frac{\partial U}{\partial q_\sigma}\delta q_\sigma - \sum_\sigma \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

$$= -\sum_\sigma \left(\frac{d}{dt}\frac{\partial (T-U)}{\partial \dot{q}_\sigma} - \frac{\partial (T-U)}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

$$L(q_\sigma, \dot{q}_\sigma; t) = T - U$$

Note : This is only true if $\frac{\partial U}{\partial \dot{q}_\sigma} = 0$

Generalized coordinates : $q_\sigma(\{x_i\})$

Define -- Lagrangian : $L \equiv T - U$

$$L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)$$

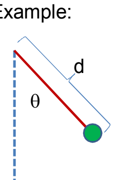
$$(\mathbf{F}\text{-}m\mathbf{a})\cdot d\mathbf{s} = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma}\right)\delta q_\sigma = 0$$

$\Rightarrow$ Minimization integral : $S = \int_{t_i}^{t_f} L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t)\,dt$

Euler – Lagrange equations : $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

Example:



$$L = L(\theta, \dot{\theta}) = \tfrac{1}{2} m d^2 \dot{\theta}^2 - mg(d - d\cos\theta)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0 \quad \Rightarrow \frac{d}{dt} m d^2 \dot{\theta} - mgd\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} = \frac{g}{d}\sin\theta$$

Another example : $L = L(\{q_\sigma\}, \{\dot{q}_\sigma\}, t) = T - U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

$$L = L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \tfrac{1}{2} I_1\left(\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2\right) + \tfrac{1}{2} I_3\left(\dot{\alpha}\cos\beta + \dot{\gamma}\right)^2 - Mgd\cos\beta$$