PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 3:

Chapter 1 – scattering theory continued; center of mass versus laboratory reference frame.

PHY 711 Classical Mechanics and Mathematical Methods

MWF 10 AM-10:50 PM	OPL 103	http://www.wfu.edu/~natalie/f13phy711/

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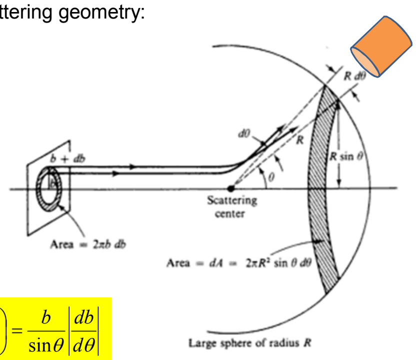
Course schedule

(Preliminary schedule -- subject to frequent adjustment.)

	Date	F&W Reading	Topic	Assignment
1	Wed, 8/28/2013	Chap. 1	Review of basic principles;Scattering theory	#1
2	Fri, 8/30/2013	Chap. 1	Scattering theory continued	#2
3	Mon, 9/02/2013	Chap. 1	Scattering theory continued	#3
4	Wed, 9/04/2013	Chap. 2	Accelerated Coordinate Systems	#4

Scattering geometry:



$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Relationship between scattering angle θ and impact parameter *b* for interaction potential *V(r)*:

$$\theta = \pi - 2b\int_{r_{\min}}^{\infty} dr \left(\frac{1/r^2}{\sqrt{1-\frac{b^2}{r^2}-\frac{V(r)}{E}}} \right)$$

where :

$$1-\frac{b^2}{r_{\min}^2}-\frac{V(r_{\min})}{E}=0$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Example of cross section analysis

Rutherford scattering : for $\frac{V(r)}{E}\equiv\frac{\kappa}{r}$

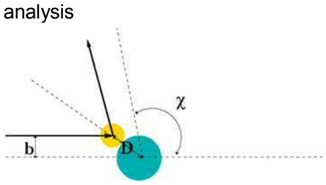
$$\theta = 2\sin^{-1}\left(\frac{1}{\sqrt{(b/\kappa)^2+1}}\right)$$

$$\frac{b}{\kappa}=\left|\frac{\cos(\theta/2)}{\sin(\theta/2)}\right|$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{\kappa^2}{4}\frac{1}{\sin^4(\theta/2)}$$

Example of cross section analysis

Hard sphere scattering:



For your homework you showed that

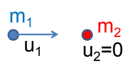
$$b = D\cos\left(\frac{\chi}{2}\right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)=\frac{b}{\sin\chi}\left|\frac{db}{d\chi}\right|=\frac{D^2}{4}$$

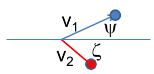
The results above were derived in the center of mass reference frame; relationship between normal laboratory reference and center of mass:

Laboratory reference frame:

Before

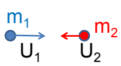


After

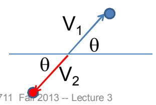


Center of mass reference frame:

Before



After



Relationship between center of mass and laboratory frames of reference

Definition of center of mass $\mathbf{R}_{CM}$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = (m_1+m_2)\mathbf{R}_{CM}$$

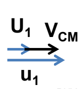
$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = (m_1+m_2)\dot{\mathbf{R}}_{CM}$$

$$m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = (m_1+m_2)\mathbf{V}_{CM} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$$

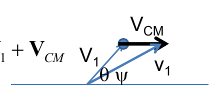
In our case :

$$\mathbf{V}_{CM} = \frac{m_1}{m_1+m_2}\mathbf{u}_1 = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1+m_2}$$

$$\mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM}$$



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$



Relationship between center of mass and laboratory frames of reference -- continued

Since m_2 is initially at rest :

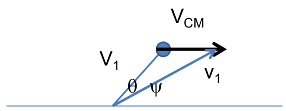
$$\mathbf{V}_{CM} = \frac{m_1}{m_1+m_2}\mathbf{u}_1 \qquad \mathbf{u}_1 = \mathbf{U}_1 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_1 = \frac{m_2}{m_1+m_2}\mathbf{u}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM}$$

$$\mathbf{u}_2 = \mathbf{U}_2 + \mathbf{V}_{CM} \quad \Rightarrow \mathbf{U}_2 = -\frac{m_1}{m_1+m_2}\mathbf{u}_1 = -\mathbf{V}_{CM}$$

$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$

$$\mathbf{v}_2 = \mathbf{V}_2 + \mathbf{V}_{CM}$$

Relationship between center of mass and laboratory frames of reference



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$
$$v_1 \sin\psi = V_1 \sin\theta$$
$$v_1 \cos\psi = V_1 \cos\theta + V_{CM}$$
$$\tan\psi = \frac{\sin\theta}{\cos\theta + V_{CM}/V_1} = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

For elastic scattering

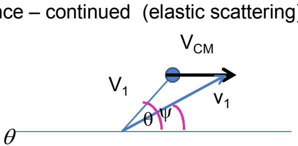
Digression – elastic scattering

$$\tfrac{1}{2}m_1U_1^2 + \tfrac{1}{2}m_2U_2^2 + \cancel{\tfrac{1}{2}(m_1+m_2)V_{CM}^2}$$
$$= \tfrac{1}{2}m_1V_1^2 + \tfrac{1}{2}m_2V_2^2 + \cancel{\tfrac{1}{2}(m_1+m_2)V_{CM}^2}$$

Also note:

$$m_1\mathbf{U}_1 + m_2\mathbf{U}_2 = 0 \qquad m_1\mathbf{V}_1 + m_2\mathbf{V}_2 = 0$$
$$\mathbf{U}_1 = \frac{m_2}{m_1}\mathbf{V}_{CM} \qquad \mathbf{U}_2 = -\mathbf{V}_{CM}$$
$$\Rightarrow |\mathbf{U}_1| = |\mathbf{V}_1| \quad \text{and} \quad |\mathbf{U}_2| = |\mathbf{V}_2| = |\mathbf{V}_{CM}|$$
$$\text{Also note that: } m_1|\mathbf{U}_1| = m_2|\mathbf{U}_2|$$
$$\text{So that: } V_{CM}/V_1 = V_{CM}/U_1 = m_1/m_2$$

Relationship between center of mass and laboratory frames of reference – continued (elastic scattering)



$$\mathbf{v}_1 = \mathbf{V}_1 + \mathbf{V}_{CM}$$
$$v_1 \sin\psi = V_1 \sin\theta$$
$$v_1 \cos\psi = V_1 \cos\theta + V_{CM}$$
$$\tan\psi = \frac{\sin\theta}{\cos\theta + V_{CM}/V_1} = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

$$\text{Also:} \quad \cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1 + 2m_1/m_2\cos\theta + (m_1/m_2)^2}}$$

Differential cross sections in different reference frames

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{d\Omega_{CM}}{d\Omega_{LAB}}$$

$$\frac{d\Omega_{CM}}{d\Omega_{LAB}} = \left|\frac{\sin\theta}{\sin\psi}\frac{d\theta}{d\psi}\right| = \left|\frac{d\cos\theta}{d\cos\psi}\right|$$

Using :

$$\cos\psi = \frac{\cos\theta + m_1/m_2}{\sqrt{1+2(m_1/m_2)\cos\theta + (m_1/m_2)^2}}$$

$$\left|\frac{d\cos\psi}{d\cos\theta}\right| = \frac{(m_1/m_2)\cos\theta + 1}{\left(1+2(m_1/m_2)\cos\theta + (m_1/m_2)^2\right)^{3/2}}$$

Differential cross sections in different reference frames – continued:

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\left|\frac{d\cos\theta}{d\cos\psi}\right|$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta + (m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta + 1}$$

$$\text{where : } \quad \tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(\theta)}{d\Omega_{CM}}\right)\frac{\left(1+2m_1/m_2\cos\theta + (m_1/m_2)^2\right)^{3/2}}{(m_1/m_2)\cos\theta + 1}$$

$$\text{where : } \quad \tan\psi = \frac{\sin\theta}{\cos\theta + m_1/m_2}$$

Example: suppose $m_1 = m_2$

$$\text{In this case : } \quad \tan\psi = \frac{\sin\theta}{\cos\theta + 1} \quad \Rightarrow \psi = \frac{\theta}{2}$$

$$\text{note that } \quad 0 \le \psi \le \frac{\pi}{2}$$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi$$

Example of cross section analysis – CM versus lab frame

Rutherford scattering : $V(r) = \frac{\kappa E}{r} = \frac{Z_1 Z_2 e^2}{r}$

(Note that E is center of mass energy)

$$\left(\frac{d\sigma}{d\Omega}\right)\bigg|_{CM} = \frac{\kappa^2}{4}\frac{1}{\sin^4(\theta/2)}$$

For $m_1 = m_2$

$$\left(\frac{d\sigma_{LAB}(\psi)}{d\Omega_{LAB}}\right) = \left(\frac{d\sigma_{CM}(2\psi)}{d\Omega_{CM}}\right)\cdot 4\cos\psi = \kappa^2\,\frac{\cos\psi}{\sin^4\psi}$$