PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

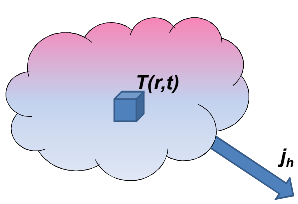
Plan for Lecture 34:

Chapter 11 in F & W:

Heat conduction

24	Wed, 10/23/2013	Chap. 5	Rigid rotations	#21
25	Fri, 10/25/2013	Chap. 5	Rigid rotations	
	Mon, 10/28/2013	No class	Take-home exam	
	Wed, 10/30/2013	No class	Take-home exam	
	Fri, 11/01/2013	No class	Take-home exam	
26	Mon, 11/04/2013	Chap. 8	Oscillations in two-dimensional membranes	Take-home exam due
27	Wed, 11/06/2013	Chap. 9	Physics of fluids	#22
28	Fri, 11/08/2013	Chap. 9	Physics of fluids	#23
29	Mon, 11/11/2013	Chap. 9	Sound Waves	#24
30	Wed, 11/13/2013	Chap. 9	Sound Waves	#25
31	Fri, 11/15/2013	Chap. 9	Non linear effects in Sound	#26
32	Mon, 11/18/2013	Chap. 10	Surface waves	preparation for presentations
33	Wed, 11/20/2013	Chap. 10	Surface waves	
34	Fri, 11/22/2013	Chap. 11	Heat conduction	
35	Mon, 11/25/2013	Chap. 12	Viscous fluids	
	Wed, 11/27/2013		*Thanksgiving Holiday*	
	Fri, 11/29/2013		*Thanksgiving Holiday*	
36	Mon, 12/02/2013		Student presentations I	
37	Wed, 12/04/2013		Student presentations II	
38	Fri, 12/06/2013		Student presentations III	
	Mon, 12/09/2013		Begin Take-home final	

Conduction of heat



Enthalpy of a system at constant pressure p

non uniform temperature $T(\mathbf{r},t)$

mass density ρ heat capacity c_p

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0$$

Conduction of heat -- continued

$$H = \int_V \rho c_p \left(T(\mathbf{r},t) - T_0\right) d^3r + H_0$$

Time rate of change of enthapy :

$$\frac{dH}{dt} = \int_V \rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} d^3r = -\int_A \mathbf{j}_h \cdot d\mathbf{A} + \int_V \rho \dot{q} d^3r$$

heat flux

heat source

$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \mathbf{j}_h + \rho \dot{q}$$

Conduction of heat -- continued

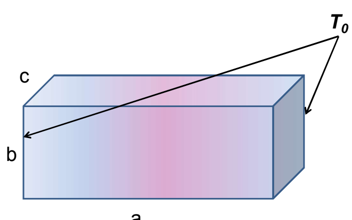
$$\rho c_p \frac{\partial T(\mathbf{r},t)}{\partial t} = -\nabla \mathbf{j}_h + \rho \dot{q}$$

Empirically : $\mathbf{j}_h = -k_{th} \nabla T(\mathbf{r},t)$

$$\Rightarrow \frac{\partial T(\mathbf{r},t)}{\partial t} = \kappa \nabla^2 T(\mathbf{r},t) + \frac{\dot{q}}{c_p}$$

$$\kappa \equiv \frac{k_{th}}{\rho c_p}$$

Boundary value problems for heat conduction

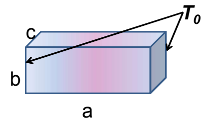


$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = \frac{\dot{q}}{c_p}$$

Without source term : $\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa \nabla^2 T(\mathbf{r},t) = 0$

Example with boundary values : $T(0,y,z,t) = T(0,y,z,t) = T_0$

Boundary value problems for heat conduction



$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(0,y,z,t) = T(0,y,z,t) = T_0$$

$$\frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,b,z,t)}{\partial y} = 0$$

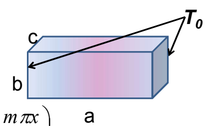
$$\frac{\partial T(x,y,0,t)}{\partial z} = \frac{\partial T(x,y,c,t)}{\partial z} = 0$$

Separation of varables : $T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$

Let $\frac{d^2X}{dx^2} = -\alpha^2 X \quad \frac{d^2Y}{dy^2} = -\beta^2 Y \quad \frac{d^2Z}{dz^2} = -\gamma^2 Z$

$$\Rightarrow -\lambda + \kappa(\alpha^2 + \beta^2 + \gamma^2) = 0$$

Boundary value problems for heat conduction



$$T(x,y,z,t) = T_0 + X(x)Y(y)Z(z)e^{-\lambda t}$$

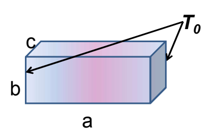
$$X(0) = X(a) = 0 \quad \Rightarrow X(x) = \sin\left(\frac{m\pi x}{a}\right)$$

$$\frac{dY(0)}{dy} = \frac{dY(b)}{dy} = 0 \quad \Rightarrow Y(y) = \cos\left(\frac{n\pi y}{b}\right)$$

$$\frac{dZ(0)}{dz} = \frac{dZ(b)}{dz} = 0 \quad \Rightarrow Z(z) = \cos\left(\frac{p\pi z}{c}\right)$$

$$-\lambda_{nmp} + \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Boundary value problems for heat conduction



Full solution :

$$T(x,y,z,t) = T_0 + \sum_{nmp} C_{nmp} \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)e^{-\lambda_{mnp}t}$$

$$\lambda_{nmp} = \kappa\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2\right) = 0$$

Initial value problem in an infinite domain; Fourier transform

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},0) = f(\mathbf{r})$$

$$\text{Let : } \quad \widetilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t)$$

$$\widetilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$\Rightarrow \widetilde{T}(\mathbf{q},0) = \widetilde{f}(\mathbf{q})$$

$$\Rightarrow \frac{\partial \widetilde{T}(\mathbf{q},t)}{\partial t} = -\kappa q^2 \widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t) = \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$\widetilde{T}(\mathbf{q},t) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} T(\mathbf{r},t) \qquad \Rightarrow T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}} \widetilde{T}(\mathbf{q},t)$$

$$\widetilde{T}(\mathbf{q},t) = \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$T(\mathbf{r},t) = \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot\mathbf{r}} \widetilde{T}(\mathbf{q},0) e^{-\kappa q^2 t}$$

$$\widetilde{T}(\mathbf{q},0) = \widetilde{f}(\mathbf{q}) = \int d^3r e^{-i\mathbf{q}\cdot\mathbf{r}} f(\mathbf{r})$$

$$T(\mathbf{r},t) = \int d^3r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with } \; G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

Initial value problem in an infinite domain; Fourier transform

$$T(\mathbf{r},t) = \int d^3r' G(\mathbf{r}-\mathbf{r}',t) T(\mathbf{r}',0)$$

$$\text{with } \; G(\mathbf{r}-\mathbf{r}',t) \equiv \frac{1}{(2\pi)^3}\int d^3q e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} e^{-\kappa q^2 t}$$

$$G(\mathbf{r}-\mathbf{r}',t) = \frac{1}{(4\pi\kappa t)^{3/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/(4\kappa t)}$$

Heat equation in half-space

$$\frac{\partial T(\mathbf{r},t)}{\partial t} - \kappa\nabla^2 T(\mathbf{r},t) = 0$$

$$T(\mathbf{r},t) \Rightarrow T(z,t) \quad \text{with initial and boundary values :}$$

$$T(z,t) \equiv 0 \quad \text{for } z < 0$$

$$T(z,0) = 0 \quad \text{for } z > 0$$

$$T(0,t) = T_0 \quad \text{for } t \geq 0$$

$$\text{Solution :} \quad T = T_0 \,\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

Heat equation in half-space -- continued

$$\frac{\partial T(z,t)}{\partial t} - \kappa \frac{\partial^2 T(z,t)}{\partial z^2} = 0$$

$$\text{Solution :} \quad T = T_0 \,\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right)$$

$$\text{where } \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

$$\text{Note that} \quad \frac{d\,\text{erfc}(x)}{dx} = \frac{d}{dx}\frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

$$\frac{\partial}{\partial t}\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)}\left(\frac{z}{4\sqrt{\kappa t^3}}\right)$$

$$\frac{\partial^2}{\partial z^2}\text{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(z^2/(4\kappa t)\right)}\left(\frac{z}{4\kappa\sqrt{\kappa t^3}}\right)$$