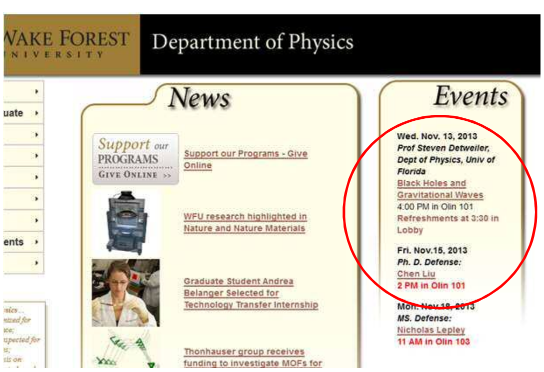
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 30:

Wave equation for sound

1. **Example of linear sound**
2. **Non-linear effects in sound**

25	Fri, 10/25/2013	Chap. 5	Rigid rotations	
	Mon, 10/28/2013	No class	Take-home exam	
	Wed, 10/30/2013	No class	Take-home exam	
	Fri, 11/01/2013	No class	Take-home exam	
26	Mon, 11/04/2013	Chap. 8	Oscillations in two-dimensional membranes	Take-home exam due
27	Wed, 11/06/2013	Chap. 9	Physics of fluids	#22
28	Fri, 11/08/2013	Chap. 9	Physics of fluids	#23
29	Mon, 11/11/2013	Chap. 9	Sound Waves	#24
0	Wed, 11/13/2013	Chap. 9	Sound Waves	#25
31	Fri, 11/15/2013			
32	Mon, 11/18/2013			
33	Wed, 11/20/2013			
34	Fri, 11/22/2013			
35	Mon, 11/25/2013			
	Wed, 11/27/2013		*Thanksgiving Holiday*	
	Fri, 11/29/2013		*Thanksgiving Holiday*	
36	Mon, 12/02/2013		Student presentations I	
37	Wed, 12/04/2013		Student presentations II	
38	Fri, 12/06/2013		Student presentations III	



WFU Physics Colloquium

TITLE: Black Holes and Gravitational Waves

SPEAKER: Professor Steven Detweiler,

Department of Physics,
University of Florida

TIME: Wednesday November 13, 2013 at 4:00

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

John Wheeler popularized the name "Black Hole" in 1967. Ever since, these unusual objects have been invoked to explain a surprisingly diverse range of phenomena. Today we apparently observe black holes with a wide variety of masses and in many different environments. However we have rather scant evidence that these objects are actually the black holes of General Relativity rather than, say, small massive object which do not obey the rules of Einstein's gravity.

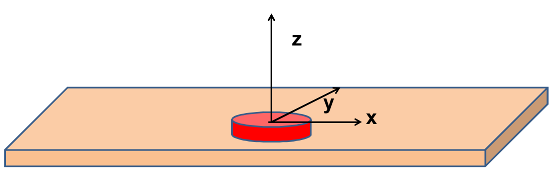
Gravitational waves provide explanations for a completely different set of phenomena. However, their direct detection continues to be a rather elusive goal. It is possible that sometime soon (five to ten years? or maybe only three years?) gravitational waves will indeed be detected, and perhaps this event will give us direct evidence that the black holes which we observe are indeed the black holes of Einstein's gravity.

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example :

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$
can be represented as boundary value of $\Phi(\mathbf{r},t)$



Define : $\tilde{\Phi}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \Phi(\mathbf{r},t) e^{i\omega t} dt$

$$\Phi(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\Phi}(\mathbf{r},\omega) e^{-i\omega t} d\omega$$

Define : $\tilde{f}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} f(\mathbf{r},t) e^{i\omega t} dt$

$$f(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\mathbf{r},\omega) e^{-i\omega t} d\omega$$

Define : $\tilde{G}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} G(\mathbf{r},t) e^{i\omega t} dt$

$$G(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(\mathbf{r},\omega) e^{-i\omega t} d\omega$$

$\tilde{G}(\mathbf{r},\omega)$ must satisfy :

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{where } k^2 = \frac{\omega^2}{c^2}$$

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V \left(h\nabla^2 g - g\nabla^2 h\right) d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot \hat{\mathbf{n}} d^2r$

$$\nabla^2\tilde{\Phi} + k^2\tilde{\Phi} = -\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

$h \leftrightarrow \tilde{\Phi}; \qquad g \leftrightarrow \tilde{G}$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot \hat{\mathbf{n}} d^2r$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega) d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot \hat{\mathbf{n}} d^2r'$$

Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) \tilde{f}(\mathbf{r}',\omega) d^3r' +$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot \hat{\mathbf{n}} d^2r'$$

Boundary values for our example :

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2 > a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2 < a^2 \end{cases}$$

Note : Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \qquad \text{where } \bar{z}' = -z'; \quad z>0$$

$$\widetilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\widetilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z>0$$

$$\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z>0$$

$$\widetilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\widetilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'\,dy'$$

$$= -i\omega\varepsilon a\int_0^a r'\,dr'\int_0^{2\pi} d\phi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain : $x' = r'\cos\phi'$

$y' = r'\sin\phi'$

For $r >> a$; $\quad |\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\phi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\phi'$$

$$\widetilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'\,dr'\int_0^{2\pi} d\phi'\,e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} d\phi'\,e^{-iu\sin\phi'} = J_0(u)$

$$\Rightarrow \widetilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'\,dr'\,J_0(kr'\sin\theta)$$

$$\int_0^w u\,du\,J_0(u) = wJ_1(w)$$

$$\Rightarrow \widetilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

Energy flux : $\mathbf{j}_e = \delta\mathbf{v}p$

Taking time average : $\langle \mathbf{j}_e \rangle = \frac{1}{2}\Re\left(\delta\mathbf{v}p^*\right)$

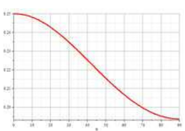
$$= \tfrac{1}{2}\rho_0\Re\left(\left(-\nabla\Phi\right)\left(-i\omega\Phi\right)^*\right)$$

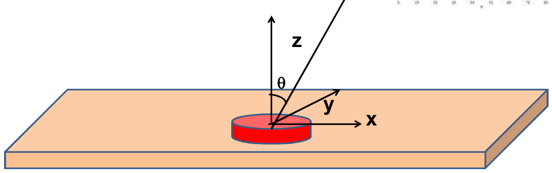
Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$

Time averaged power per solid angle :

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle \mathbf{j}_e \rangle \cdot \hat{\mathbf{r}} r^2 = \frac{1}{2}\rho_0 \varepsilon^2 c^3 k^4 a^6 \left| \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right|^2$$





Effects of nonlinearities in fluid equations
-- one dimensional case

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{v}\right) = 0$

Assume that spatial variation confined to x direction ; assume that $\mathbf{v} = v\hat{\mathbf{x}}$ and $\mathbf{f}_{applied} = 0$.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0$$
$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0$$

Expressing p in terms of ρ: $p = p(\rho)$

$$\frac{\partial p}{\partial x}=\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x}\equiv c^2(\rho)\frac{\partial \rho}{\partial x} \qquad \text{where } \frac{\partial p}{\partial \rho}\equiv c^2(\rho)$$

For adiabatic ideal gas:

$$c^2(\rho)=\frac{\gamma p}{\rho}=c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} \qquad \text{where } c_0^2\equiv\frac{\gamma p_0}{\rho_0}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$
$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial x}=0$$

Expressing variation v in terms of variation of ρ:

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t}+v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$
$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}=0$$

Traveling wave solution:

Assume: $\rho=\rho_0+f(x-u(\rho)t)$

Need to find self-consistent equations for propagation velocity $u(\rho)$ using equations

$$\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial t}+v\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}=0$$
$$\frac{\partial \rho}{\partial t}+v\frac{\partial \rho}{\partial x}+\rho\frac{\partial v}{\partial \rho}\frac{\partial \rho}{\partial x}=0$$