PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 29 --

Chap. 9: Wave equation for sound

1. **Standing waves**
2. **Green's function for wave equation; wave scattering**

25	Fri, 10/25/2013	Chap. 5	Rigid rotations	
	Mon, 10/28/2013	No class	Take-home exam	
	Wed, 10/30/2013	No class	Take-home exam	
	Fri, 11/01/2013	No class	Take-home exam	
26	Mon, 11/04/2013	Chap. 8	Oscillations in two-dimensional membranes	Take-home exam due
27	Wed, 11/06/2013	Chap. 9	Physics of fluids	#22
28	Fri, 11/08/2013	Chap. 9	Physics of fluids	#23
29	Mon, 11/11/2013	Chap. 9	Sound Waves	#24
30	Wed, 11/13/2013			
31	Fri, 11/15/2013			
32	Mon, 11/18/2013			
33	Wed, 11/20/2013			
34	Fri, 11/22/2013			
35	Mon, 11/25/2013			
	Wed, 11/27/2013		*Thanksgiving Holiday*	
	Fri, 11/29/2013		*Thanksgiving Holiday*	
36	Mon, 12/02/2013		Student presentations I	
37	Wed, 12/04/2013		Student presentations II	
38	Fri, 12/06/2013		Student presentations III	

WFU Physics Ph. D. Thesis Defense

TITLE: Measuring the Microscale Mechanical Properties of Fibrin Fibers and Cancer Cells

SPEAKER: Justin Sigley,

Department of Physics
Wake Forest University

TIME: Monday November 11, 2013 at 11 AM

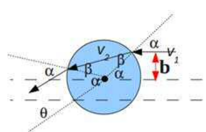
PLACE: Room 103 Olin Physical Laboratory

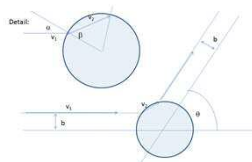
All interested persons are cordially invited to attend.

ABSTRACT

The microscale material properties dictate the macroscale behavior of biological systems. Fibrinogen, one of the most abundant proteins in the blood, is converted into fibrin fibers that perform the essential mechanical task of stemming the flow of blood. Fibrinogen fibers can be fabricated by a technique called electrospinning. We studied the mechanical properties of dry, electrospun fibrinogen fibers using a combined atomic force/fluorescence microscopy technique. The mechanical properties of these electrospun fibers is important

Comments about exam





Note that this figure is misleading; α>β for physical case

$$\theta = 2\alpha - 2\beta$$

$$v_1 \sin\alpha = v_2 \sin\beta$$

$$\frac{\sin^2\alpha}{\sin^2\beta} = 1 + \frac{U_0}{E} \equiv n^2$$

$$b = a\sin\alpha = \frac{na\sin(\theta/2)}{\sqrt{n^2 - 2a\cos(\theta/2) + 1}}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{1}{2\sin\theta}\left|\frac{db^2}{d\theta}\right|$$

$$\sigma_T = \int d\Omega \frac{d\sigma}{d\Omega} = \pi[b(\theta)]^2\Big|_{\theta=0}^{\theta=\theta_m}$$

Comment about exam – continued

$$H = \frac{p^2}{2s^2m} + \frac{p_s^2}{2\varepsilon} + V(q) + gkT\ln s$$

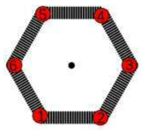
$$\frac{dp_s}{dt} = \varepsilon\frac{d^2s}{dt^2} = \frac{p^2}{s^2m} - \frac{gkT}{s}$$

$$\text{Note that: } \frac{dp_s}{dt} = 0 \quad \text{for} \quad s \equiv s_0 = \frac{p^2}{mgkT}$$

$$\text{Suppose that: } s(t) = s_0 + s_1(t)$$

$$\varepsilon\frac{d^2s_1}{dt^2} = \frac{p^2}{(s_0+s_1)^2 m} - \frac{gkT}{(s_0+s_1)} \approx -\left(\frac{p^2}{s_0^2 m}\right)s_1$$

Comment about exam – continued



Linearization of the fluid dynamics relations:

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$

Near equilibrium :

$$\rho = \rho_0 + \delta\rho$$

$$p = p_0 + \delta p$$

$$\mathbf{v} = 0 + \delta\mathbf{v}$$

$$\mathbf{f}_{applied} = 0$$

Equations to lowest order in perturbation :

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\mathbf{v} = 0$$

Velocity potential : $\delta\mathbf{v} = -\nabla\Phi$

Pressure in terms of the density :

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_{s,\rho_0,p_0} \delta\rho \equiv c^2 \delta\rho$$

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Wave equation for air :

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p_0}{\rho_0}$

$\mathbf{v} = -\nabla\Phi$

Boundary values :

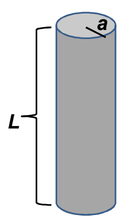
Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}} \cdot \mathbf{V} = \hat{\mathbf{n}} \cdot \delta\mathbf{v} = -\hat{\mathbf{n}} \cdot \nabla\Phi$$

Free surface :

$$\delta p = 0 \qquad \Rightarrow \rho_0 \frac{\partial \Phi}{\partial t} = 0$$

Time harmonic standing waves in a pipe



$$\frac{\partial^2 \Phi}{\partial t^2} - c^2 \nabla^2 \Phi = 0$$

Boundary values :

At fixed surface : $\hat{\mathbf{n}} \cdot \nabla\Phi = 0$

At free surface : $\frac{\partial \Phi}{\partial t} = 0$

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \qquad \text{Define :} \quad k \equiv \frac{\omega}{c}$$

In cylindrical coordinates :

$$\Phi(r,\varphi,z,t) = R(r)F(\varphi)Z(z)e^{-i\omega t} \equiv R(r)F(\varphi)Z(z)e^{-ikct}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\Phi(r,\varphi,z,t) = 0$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}+k^2\right)\Phi(r,\varphi,z,t)=0$$

$$\Phi(r,\varphi,z,t)=R(r)F(\varphi)Z(z)e^{-i\omega t}$$

$$F(\varphi)=e^{im\varphi};\ F(\varphi)=F(\varphi+2\pi N)\Rightarrow m=\text{integer}$$

$$Z(z)=e^{i\alpha z};\ \alpha=\text{real plus other restrictions}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\alpha^2+k^2\right)R(r)=0$$

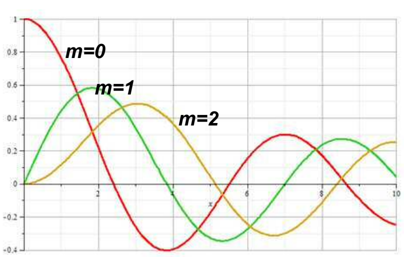
For $k^2\geq\alpha^2$ define $\kappa^2\equiv k^2-\alpha^2$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}+\kappa^2\right)R(r)=0$$

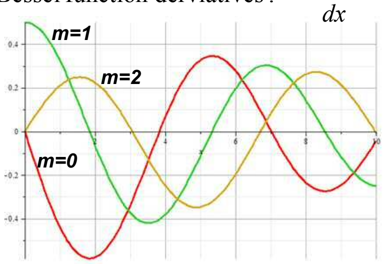
Cylinder surface boundary conditions : $\left.\frac{dR}{dr}\right|_{r=a}=0$

$$\Rightarrow R(r)=J_m(\kappa r)\quad\text{where for}\quad\frac{dJ_m(x'_{mn})}{dx}=0,\ \kappa_{mn}=\frac{x'_{mn}}{a}$$

Bessel functions : $J_m(x)$



Bessel function derviatives : $\frac{dJ_m(x)}{dx}$



Zeros of derivatives: m=0: 0.00000, 3.83171, 7.01559
m=1: 1.84118, 5.33144, 8.53632
m=2: 3.05424, 6.70613, 9.96947

Boundary condition for z=0, z=L:

For open - open pipe :

$$Z(0) = Z(L) = 0 \qquad \Rightarrow Z(z) = \sin\left(\frac{p\pi z}{L}\right)$$

$$\Rightarrow \alpha_p = \frac{p\pi}{L}, \qquad p = 1,2,3...$$

Resonant frequencies :

$$\frac{\omega^2}{c^2} = k^2 = \kappa_{mn}^2 + \alpha_p^2$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2$$

Example

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi p}{L}\right)^2 = \left(\frac{\pi p}{L}\right)^2\left(1 + \left(\frac{L}{a}\right)^2\left(\frac{x'_{mn}}{\pi p}\right)^2\right)$$

$$\pi p = 3.14, 6.28, 9.42....$$

$$x'_{mn} = 0.00, 1.84, 3.05$$

Alternate boundary condition for z=0, z=L:

For open - closed pipe :

$$\frac{dZ(0)}{dz} = Z(L) = 0 \quad \Rightarrow Z(z) = \cos\left(\frac{(2p+1)\pi z}{2L}\right)$$

$$\Rightarrow \alpha_p = \frac{(2p+1)\pi}{2L}, \quad p = 0,1,2,3...$$

$$k_{mnp}^2 = \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{\pi(2p+1)}{2L}\right)^2$$

Other solutions to wave equation:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0$$

Plane wave solution :

$$\Phi(\mathbf{r},t) = Ae^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} \quad \text{where} \quad k^2 = \left(\frac{\omega}{c}\right)^2$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Solution in terms of Green's function :

$$\Phi(\mathbf{r},t) = \int d^3r' \int dt' \, G(\mathbf{r}-\mathbf{r}', t-t') f(\mathbf{r}',t')$$

where

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Wave equation with source -- continued:

We can show that :

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{\delta\left(t'-\left(t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

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Derivation of Green's function for wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}',t-t') = -\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Recall that

$$\delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t')}d\omega$$

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Derivation of Green's function for wave equation -- continued

Define : $\tilde{G}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} G(\mathbf{r},t)e^{i\omega t}dt$

$$G(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

$\tilde{G}(\mathbf{r},\omega)$ must satisfy :

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{where } k^2 = \frac{\omega^2}{c^2}$$

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Derivation of Green's function for wave equation -- continued

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

Solution assuming isotropy in $\mathbf{r}-\mathbf{r}'$:

$$\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Check -- Define $R \equiv |\mathbf{r}-\mathbf{r}'|$ and for $R > 0$:

$$\left(\nabla^2 + k^2\right)\tilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\tilde{G}(R,\omega) = 0$$

Derivation of Green's function for wave equation -- continued

For $R > 0$:

$$\left(\nabla^2 + k^2\right)\tilde{G}(R,\omega) = \frac{1}{R}\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\tilde{G}(R,\omega) = 0$$

$$\frac{d^2}{dR^2}\left(R\tilde{G}(R,\omega)\right) + k^2\left(R\tilde{G}(R,\omega)\right) = 0$$

$$\left(R\tilde{G}(R,\omega)\right) = A\, e^{ikR} + Be^{-ikR}$$

$$\Rightarrow \tilde{G}(R,\omega) = A\frac{e^{ikR}}{R} + B\frac{e^{-ikR}}{R}$$

Derivation of Green's function for wave equation – continued
need to find *A* and *B*.

Note that: $\nabla^2 \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\delta(\mathbf{r}-\mathbf{r}')$

$$\Rightarrow A = B = \frac{1}{4\pi}$$

$$\tilde{G}(R,\omega) = \frac{e^{\pm ikR}}{4\pi R}$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{G}(\mathbf{r}-\mathbf{r}',\omega)e^{-i\omega(t-t')}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

Derivation of Green's function for wave equation – continued

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}e^{-i\omega(t-t')}d\omega$$

Noting that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega u}d\omega = \delta(u)$

$$\Rightarrow G(\mathbf{r}-\mathbf{r}',t-t') = \frac{\delta\left(t-\left(t'\mp\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

For time harmonic forcing term we can use the corresponding Green's function:

$$\widetilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

In fact, this Green's function is appropriate for boundary conditions at infinity. For surface boundary conditions where we know the boundary values or their gradients, the Green's function must be modified.

Green's theorem

Consider two functions $h(\mathbf{r})$ and $g(\mathbf{r})$

Note that : $\int_V \left(h\nabla^2 g - g\nabla^2 h\right) d^3r = \oint_S \left(h\nabla g - g\nabla h\right)\cdot \hat{\mathbf{n}} d^2r$

$$\nabla^2\tilde{\Phi} + k^2\tilde{\Phi} = -\tilde{f}(\mathbf{r},\omega)$$

$$\left(\nabla^2 + k^2\right)\tilde{G}(\mathbf{r}-\mathbf{r}',\omega) = -\delta(\mathbf{r}-\mathbf{r}')$$

$$h \leftrightarrow \tilde{\Phi}; \qquad g \leftrightarrow \tilde{G}$$

$$\int_V \left(\tilde{\Phi}(\mathbf{r},\omega)\delta(\mathbf{r}-\mathbf{r}') - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)f(\mathbf{r},\omega)\right) d^3r =$$

$$\oint_S \left(\tilde{\Phi}(\mathbf{r},\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r},\omega)\right)\cdot\hat{\mathbf{n}} d^2r$$

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) f(\mathbf{r}',\omega) d^3r' +$$

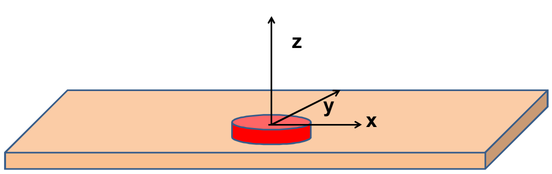
$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}} d^2r'$$

Wave equation with source:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -f(\mathbf{r},t)$$

Example :

$f(\mathbf{r},t) \Rightarrow$ time harmonic piston of radius a, amplitude $\varepsilon\hat{\mathbf{z}}$
can be represented as boundary value of $\Phi(\mathbf{r},t)$



Treatment of boundary values for time-harmonic force:

$$\tilde{\Phi}(\mathbf{r},\omega) = \int_V \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\tilde{f}(\mathbf{r}',\omega)d^3r' +$$
$$\oint_S \left(\tilde{\Phi}(\mathbf{r}',\omega)\nabla'\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) - \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\nabla'\tilde{\Phi}(\mathbf{r}',\omega)\right)\cdot\hat{\mathbf{n}}d^2r'$$

Boundary values for our example :

$$\left(\frac{\partial\tilde{\Phi}}{\partial z}\right)_{z=0} = \begin{cases} 0 & \text{for } x^2+y^2 > a^2 \\ i\omega\varepsilon a & \text{for } x^2+y^2 < a^2 \end{cases}$$

Note : Need Green's function with vanishing gradient at $z=0$:

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z>0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'dy'$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{e^{ik|\mathbf{r}-\bar{\mathbf{r}}'|}}{4\pi|\mathbf{r}-\bar{\mathbf{r}}'|} \quad \text{where } \bar{z}' = -z'; \quad z>0$$

$$\tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)_{z'=0} = \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}; \quad z>0$$

$$\tilde{\Phi}(\mathbf{r},\omega) = -\oint_{S:\,z'=0} \tilde{G}(|\mathbf{r}-\mathbf{r}'|,\omega)\frac{\partial\tilde{\Phi}(\mathbf{r}',\omega)}{\partial z}dx'dy'$$
$$= -i\omega\varepsilon a\int_0^a r'dr'\int_0^{2\pi} d\phi' \left.\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{2\pi|\mathbf{r}-\mathbf{r}'|}\right|_{z'=0}$$

Integration domain : $x' = r'\cos\phi'$
$y' = r'\sin\phi'$

For $r >> a$; $|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}'$

Assume $\hat{\mathbf{r}}$ is in the yz plane; $\phi = \frac{\pi}{2}$

$$\hat{\mathbf{r}} = \sin\theta\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$
$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{r}}\cdot\mathbf{r}' = r - r'\sin\theta\sin\phi'$$

$$\widetilde{\Phi}(\mathbf{r},\omega) = -\frac{i\omega\varepsilon a}{2\pi}\frac{e^{ikr}}{r}\int_0^a r'\,dr'\int_0^{2\pi} d\phi'\, e^{-ikr'\sin\theta\sin\phi'}$$

Note that : $\frac{1}{2\pi}\int_0^{2\pi} d\phi'\, e^{-iu\sin\phi'} = J_0(u)$

$$\Rightarrow \widetilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a\frac{e^{ikr}}{r}\int_0^a r'\,dr'\,J_0(kr'\sin\theta)$$

$$\int_0^w u\,du\,J_0(u) = wJ_1(w)$$

$$\Rightarrow \widetilde{\Phi}(\mathbf{r},\omega) = -i\omega\varepsilon a^3\frac{e^{ikr}}{r}\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

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Energy flux : $\mathbf{j}_e = \delta\mathbf{v}p$

Taking time average : $\langle\mathbf{j}_e\rangle = \frac{1}{2}\Re\left(\delta\mathbf{v}p^*\right)$

$$= \tfrac{1}{2}\rho_0\Re\left((-\nabla\Phi)(-i\omega\Phi)^*\right)$$

Time averaged power per solid angle :

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \langle\mathbf{j}_e\rangle\cdot\hat{\mathbf{r}}r^2 = \frac{1}{2}\rho_0\varepsilon^2c^3k^4a^6\left|\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right|^2$$

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