PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 28: Chap. 9 of F&W

Introduction to hydrodynamics

1. **Details of Euler formulation of hydrodynamic equations**
2. **Bernoulli integrals for irrotational flow**
3. **Sound equations**

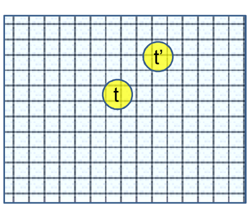
13	Wed, 9/25/2013	Chap. 3 & 6	Hamiltonian formalism	#12
14	Fri, 9/27/2013	Chap. 3 & 6	Hamiltonian formalism	#13
15	Mon, 9/30/2013	Chap. 4	Small Oscillations	#14
16	Wed, 10/02/2013	Chap. 4	Small Oscillations	
17	Fri, 10/04/2013	Chap. 4	Small Oscillations	#15
18	Mon, 10/07/2013	Chap. 4 & 7	Small Oscillations and waves	#16
19	Wed, 10/09/2013	Chap. 7	Wave equation	
	Fri, 10/11/2013		No class (Fall Break)	
20	Mon, 10/14/2013	Chap. 7	Wave equation (Presentation topic due)	#17
21	Wed, 10/16/2013	Chap. 7	Mathematical methods	#18
22	Fri, 10/18/2013	Chap. 7	Mathematical methods	#19
23	Mon, 10/21/2013	Chap. 5	Rigid rotations	#20
24	Wed, 10/23/2013	Chap. 5	Rigid rotations	#21
25	Fri, 10/25/2013	Chap. 5	Rigid rotations	
	Mon, 10/28/2013	No class	Take-home exam	
	Wed, 10/30/2013	No class	Take-home exam	
	Fri, 11/01/2013	No class	Take-home exam	
26	Mon, 11/04/2013	Chap. 8	Oscillations in two-dimensional membranes	Take-home exam due
27	Wed, 11/06/2013	Chap. 9	Physics of fluids	#22
28	Fri, 11/08/2013	Chap. 9	Physics of fluids	#23

Newton's equations for fluids
Use **Euler** formulation; properties described in terms of stationary spatial grid

Variables: Density $\rho(x,y,z,t)$

Pressure $p(x,y,z,t)$

Velocity $\mathbf{v}(x,y,z,t)$



Particle at t: $\mathbf{r}, t$

Particle at t': $\mathbf{r} + \mathbf{v}\delta t, t'$

$$t' = t + \delta t$$

Euler analysis -- continued

$$\text{Particle at } t: \quad \mathbf{r}, t$$
$$\text{Particle at } t': \quad \mathbf{r}+\mathbf{v}\delta t, t' \qquad \text{where} \quad \delta t = t' - t$$
$$\text{For} \quad f(\mathbf{r},t):$$
$$\frac{df}{dt} = \lim_{\delta t \to 0}\left(\frac{f(\mathbf{r},t') - f(\mathbf{r},t)}{\delta t} + \frac{f(\mathbf{r}+\mathbf{v}\delta t, t) - f(\mathbf{r},t)}{\delta t} \right)$$
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v}\cdot\nabla) f$$

11/08/2013 PHY 711 Fall 2013 -- Lecture 28 4

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbf{v}) = 0$$
$$\frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\mathbf{v}) + (\nabla\rho)\cdot\mathbf{v} = 0$$
$$\text{Consider}: \quad \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla\rho)\cdot\mathbf{v}$$
$$\Rightarrow \frac{d\rho}{dt} + (\rho\nabla)\cdot\mathbf{v} = 0 \qquad \text{alternative form of continuity equation}$$

11/08/2013 PHY 711 Fall 2013 -- Lecture 28 5

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v}) = 0$ "irrotational flow"
   $\Rightarrow \mathbf{v} = -\nabla\Phi$
2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force
3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$
$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

11/08/2013 PHY 711 Fall 2013 -- Lecture 28 6

Bernoulli's integral of Euler's equation for constant ρ

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C(t)\big)$

$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C_0 \qquad \text{Bernoulli's theorem}$$

For incompressible fluid

Solution of Euler's equation for fluids -- isentropic

$$\frac{\partial\mathbf{v}}{\partial t}+\nabla\left(\tfrac{1}{2}v^2\right)-\mathbf{v}\times\left(\nabla\times\mathbf{v}\right)=\mathbf{f}_{applied}-\frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v})=0$ "irrotational flow"

   $\Rightarrow\mathbf{v}=-\nabla\Phi$

2. $\mathbf{f}_{applied}=-\nabla U$ conservative applied force

3. $\rho\neq$ (constant) isentropic fluid

A little thermodynamics

First law of thermodynamics: $dE_{\text{int}}=dQ-dW$

For isentropic conditions: $dQ=0$

$dE_{\text{int}}=-dW=-pdV$

Solution of Euler's equation for fluids – isentropic (continued)

$dE_{\text{int}}=-dW=pdV$

In terms of mass density: $\rho=\frac{M}{V}$

For fixed M and variable V: $d\rho=-\frac{M}{V^2}dV$

$$dV=-\frac{M}{\rho^2}d\rho$$

In terms in intensive variables: Let $E_{\text{int}}=M\varepsilon$

$$dE_{\text{int}}=Md\varepsilon=-dW=-pdV=M\frac{p}{\rho^2}d\rho$$

$$d\varepsilon=\frac{p}{\rho^2}d\rho \qquad \left(\frac{\partial\varepsilon}{\partial\rho}\right)_{dQ=0}=\frac{p}{\rho^2}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} = \frac{p}{\rho^2}$$

$$\text{Consider:} \quad \nabla\varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho}\right)_{dQ=0} \nabla\rho = \frac{p}{\rho^2}\nabla\rho$$

$$\text{Rearranging:} \quad \nabla\left(\varepsilon + \frac{p}{\rho}\right) = \frac{\nabla p}{\rho}$$

Solution of Euler's equation for fluids – isentropic (continued)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times\left(\nabla\times\mathbf{v}\right) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

$$\frac{\nabla p}{\rho} = \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\nabla\times\mathbf{v} = 0 \qquad \mathbf{v} = -\nabla\Phi \qquad \mathbf{f}_{applied} = -\nabla U$$

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \nabla\left(\varepsilon + \frac{p}{\rho}\right)$$

$$\Rightarrow \nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Summary of Bernoulli's results

For incompressible fluid

$$\nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

For isentropic fluid with internal energy density ε

$$\nabla\left(\varepsilon + \frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Application of fluid equations to the case of air in equilibrium plus small perturbation

Newton - Euler equation of motion :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation : $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$

Near equilibrium :

$$\rho = \rho_0 + \delta\rho$$
$$p = p_0 + \delta p$$
$$\mathbf{v} = 0 + \delta\mathbf{v}$$
$$\mathbf{f}_{applied} = 0$$

Equations to lowest order in perturbation :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f}_{applied} - \frac{\nabla p}{\rho} \quad\Rightarrow\quad \frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \quad\Rightarrow\quad \frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{v} = 0$$

In terms of the velocity potential :

$$\delta\mathbf{v} = -\nabla\Phi$$

$$\frac{\partial \delta\mathbf{v}}{\partial t} = -\frac{\nabla \delta p}{\rho_0} \quad\Rightarrow \nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{v} = 0 \quad\Rightarrow \frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0$$

Expressing pressure in terms of the density :

$p = p(s,\rho) = p_0 + \delta p$ where s denotes the (constant) entropy

$p_0 = p(s,\rho_0)$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho \equiv c^2\delta\rho$$

$$\nabla\left(-\frac{\partial\Phi}{\partial t} + \frac{\delta p}{\rho_0}\right) = 0 \quad\Rightarrow -\frac{\partial\Phi}{\partial t} + c^2\frac{\delta\rho}{\rho_0} = (\text{constant})$$

$$\Rightarrow -\frac{\partial^2\Phi}{\partial t^2} + \frac{c^2}{\rho_0}\frac{\partial\delta\rho}{\partial t} = 0$$

$$\frac{\partial \delta\rho}{\partial t} - \rho_0\nabla^2\Phi = 0 \quad\Rightarrow \frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Wave equation for air:

$$\frac{\partial^2\Phi}{\partial t^2} - c^2\nabla^2\Phi = 0$$

Here, $c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$

$\mathbf{v} = -\nabla\Phi$

Note that, we also have:

$$\frac{\partial^2\delta\rho}{\partial t^2} - c^2\nabla^2\delta\rho = 0$$

$$\frac{\partial^2\delta p}{\partial t^2} - c^2\nabla^2\delta p = 0$$

Boundary values:

Impenetrable surface with normal $\hat{\mathbf{n}}$ moving at velocity $\mathbf{V}$:

$$\hat{\mathbf{n}}\cdot\mathbf{V} = \hat{\mathbf{n}}\cdot\delta\mathbf{v} = -\hat{\mathbf{n}}\cdot\nabla\Phi$$

Free surface:

$$\delta p = 0 \quad \Rightarrow \rho_0\frac{\partial\Phi}{\partial t} = 0$$

11/08/2013 PHY 711 Fall 2013 -- Lecture 28 16

Analysis of wave velocity in an ideal gas:

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

Equation of state for ideal gas:

$$pV = NkT \qquad N = \frac{M}{M_0}$$

$$p = \frac{M}{V}\frac{k}{M_0}T = \rho\frac{k}{M_0}T$$

$k = 1.38\times10^{-23}\, J/k$

M_0 = average mass of each molecule

11/08/2013 PHY 711 Fall 2013 -- Lecture 28 17

Internal energy for ideal gas:

$$E = \frac{f}{2}NkT = M\varepsilon \qquad \varepsilon = \frac{f}{2}\frac{k}{M_0}T = \frac{f}{2}\frac{p}{\rho}$$

In terms of specific heat ratio: $\gamma \equiv \frac{C_p}{C_V}$

$$dE = dQ - dW$$

$$C_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{f}{2}\frac{Mk}{M_0}$$

$$C_p = \left(\frac{dQ}{dT}\right)_p = \left(\frac{\partial E}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \frac{f}{2}\frac{Mk}{M_0} + \frac{Mk}{M_0}$$

$$\gamma = \frac{C_p}{C_V} = \frac{\frac{f}{2}+1}{\frac{f}{2}} \qquad \Rightarrow \frac{f}{2} = \frac{1}{\gamma - 1}$$

11/08/2013 PHY 711 Fall 2013 -- Lecture 28 18

Internal energy for ideal gas :

$$E = \frac{1}{\gamma - 1} NkT = M\varepsilon \qquad \varepsilon = \frac{1}{\gamma - 1}\frac{k}{M_0}T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Internal energy for ideal gas under isentropic conditions :

$$d\varepsilon = -\frac{p}{M}dV = \frac{p}{\rho^2}d\rho$$

$$\left(\frac{\partial \varepsilon}{\partial \rho}\right)_s = \frac{p}{\rho^2} = \frac{\partial}{\partial \rho}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)_s = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{1}{(\gamma - 1)\rho} - \frac{p}{(\gamma - 1)\rho^2}$$

$$\Rightarrow \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

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Alternative derivation :

Isentropic or adiabatic equation of state :

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \qquad \Rightarrow \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

Linearized speed of sound

$$c_0{}^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s, p_0, \rho_0} = \frac{p_0\gamma}{\rho_0}$$

$$c_0{}^2 \approx \frac{1.5 \cdot 1.013 \times 10^5\, Pa}{1.3 kg/m^3} \qquad c_0 \approx 340 \text{ m/s}$$

11/08/2013 PHY 711 Fall 2013 -- Lecture 28 20

Density dependence of speed of sound for ideal gas :

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{p\gamma}{\rho}$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$c^2 = \frac{p_0\gamma}{\rho_0}\frac{p/p_0}{\rho/\rho_0} = c_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}$$

11/08/2013 PHY 711 Fall 2013 -- Lecture 28 21