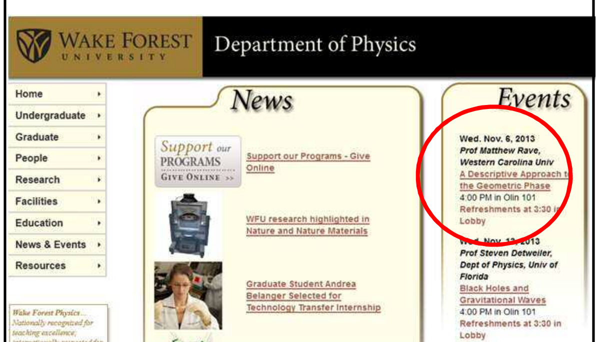
PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 27:

Introduction to hydrodynamics

1. Motivation for topic
2. Newton's laws for fluids
3. Conservation relations

13	Wed, 9/25/2013	Chap. 3 & 6	Hamiltonian formalism	#12
14	Fri, 9/27/2013	Chap. 3 & 6	Hamiltonian formalism	#13
15	Mon, 9/30/2013	Chap. 4	Small Oscillations	#14
16	Wed, 10/02/2013	Chap. 4	Small Oscillations	
17	Fri, 10/04/2013	Chap. 4	Small Oscillations	#15
18	Mon, 10/07/2013	Chap. 4 & 7	Small Oscillations and waves	#16
19	Wed, 10/09/2013	Chap. 7	Wave equation	
	Fri, 10/11/2013		No class (Fall Break)	
20	Mon, 10/14/2013	Chap. 7	Wave equation (Presentation topic due)	#17
21	Wed, 10/16/2013	Chap. 7	Mathematical methods	#18
22	Fri, 10/18/2013	Chap. 7	Mathematical methods	#19
23	Mon, 10/21/2013	Chap. 5	Rigid rotations	#20
24	Wed, 10/23/2013	Chap. 5	Rigid rotations	#21
25	Fri, 10/25/2013	Chap. 5	Rigid rotations	
	Mon, 10/28/2013	No class	Take-home exam	
	Wed, 10/30/2013	No class	Take-home exam	
	Fri, 11/01/2013	No class	Take-home exam	
26	Mon, 11/04/2013	Chap. 8	Oscillations in two-dimensional membranes	Take-home exam due
27	Wed, 11/06/2013	Chap. 9	Physics of fluids	#22



WFU Physics Colloquium

TITLE: A Descriptive Approach to the Geometric Phase

SPEAKER: Professor Matthew Rave,

Department of Physics
Western Carolina University

TIME: Wednesday November 6, 2013 at 4:00 PM

PLACE: Room 101 Olin Physical Laboratory

Refreshments will be served at 3:30 PM in the Olin Lounge. All interested persons are cordially invited to attend.

ABSTRACT

What do Möbius strips, a Chopin étude, Spirographs, Disneyland's Mad Tea Party ride, and quantum mechanics all have in common? In revealing the answer, we will investigate the geometric phase, which can be visualized as the interplay between the two characteristic periods of a closed orbit which go in and out of "synch". In this talk we will use several simple mechanical systems to illustrate two approaches to determining geometric phase: direct computation from the equations of motion, and the use of conservation laws. The elegant simplicity of this last approach can be explained by observing invariants under an action of the circle group on the torus. We conclude by describing in brief how the conservation law approach extrapolates to the general method of reduction by symmetry.

Motivation

1. Natural progression from strings, membranes, fluids; description of 1, 2, and 3 dimensional continua
2. Interesting and technologically important phenomena associated with fluids

Plan

1. Newton's laws for fluids
2. Continuity equation
3. Stress tensor
4. Energy relations
5. Bernoulli's theorem
6. Various examples
7. Sound waves

Newton's equations for fluids

Use Lagrange formulation; following "particles" of fluid

$$\text{Variables:} \quad \text{Density} \quad \rho(x,y,z,t)$$

$$\text{Pressure} \quad p(x,y,z,t)$$

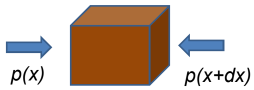
$$\text{Velocity} \quad \mathbf{v}(x,y,z,t)$$

$$m\mathbf{a} = \mathbf{F}$$

$$m \to \rho dV$$

$$\mathbf{a} \to \frac{d\mathbf{v}}{dt}$$

$$\mathbf{F} \to \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$



$$F_{pressure}\Big|_x = \left(-p(x+dx,y,z)+p(x,y,z)\right)dydz$$
$$= \frac{\left(-p(x+dx,y,z)+p(x,y,z)\right)}{dx}dxdydz$$
$$= -\frac{\partial p}{\partial x}dV$$

Newton's equations for fluids -- continued

$$m\mathbf{a} = \mathbf{F}_{applied} + \mathbf{F}_{pressure}$$
$$\rho dV\frac{d\mathbf{v}}{dt} = \mathbf{f}_{applied}\,\rho dV - (\nabla p)dV$$
$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{f}_{applied} - \nabla p$$

Detailed analysis of acceleration term :

$$\mathbf{v} = \mathbf{v}(x,y,z,t)$$
$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial\mathbf{v}}{\partial y}\frac{dy}{dt} + \frac{\partial\mathbf{v}}{\partial z}\frac{dz}{dt} + \frac{\partial\mathbf{v}}{\partial t}$$
$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial x}v_x + \frac{\partial\mathbf{v}}{\partial y}v_y + \frac{\partial\mathbf{v}}{\partial z}v_z + \frac{\partial\mathbf{v}}{\partial t}$$
$$\frac{d\mathbf{v}}{dt} = (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\partial\mathbf{v}}{\partial t}$$

Note that :

$$\frac{\partial\mathbf{v}}{\partial x}v_x + \frac{\partial\mathbf{v}}{\partial y}v_y + \frac{\partial\mathbf{v}}{\partial z}v_z = \nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times(\nabla\times\mathbf{v})$$

Newton's equations for fluids -- continued

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f}_{applied} - \nabla p$$

$$\rho\left(\nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) + \frac{\partial \mathbf{v}}{\partial t}\right) = \rho \mathbf{f}_{applied} - \nabla p$$

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\mathbf{v}) + (\nabla\rho)\cdot\mathbf{v} = 0$$

$$\text{Consider: } \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + (\nabla\rho)\cdot\mathbf{v}$$

$$\Rightarrow \frac{d\rho}{dt} + \rho(\nabla\cdot\mathbf{v}) = 0$$

alternative form of continuity equation

Solution of Euler's equation for fluids

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v}) = \mathbf{f}_{applied} - \frac{\nabla p}{\rho}$$

Consider the following restrictions:

1. $(\nabla\times\mathbf{v}) = 0$ "irrotational flow"

   $\Rightarrow \mathbf{v} = -\nabla\Phi$

2. $\mathbf{f}_{applied} = -\nabla U$ conservative applied force

3. $\rho = \text{(constant)}$ incompressible fluid

$$\frac{\partial(-\nabla\Phi)}{\partial t} + \nabla\left(\tfrac{1}{2}v^2\right) = -\nabla U - \frac{\nabla p}{\rho}$$

$$\Rightarrow \nabla\left(\frac{p}{\rho} + U + \tfrac{1}{2}v^2 - \frac{\partial\Phi}{\partial t}\right) = 0$$

Bernoulli's integral of Euler's equation

$$\nabla\left(\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}\right)=0$$

Integrating over space:

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=C(t)$$

where $\mathbf{v}=-\nabla\Phi(\mathbf{r},t)=-\nabla\big(\Phi(\mathbf{r},t)+C'(t)\big)$

$$\Rightarrow\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0 \qquad \text{Bernoulli's theorem}$$

Examples of Bernoulli's theorem

$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2-\frac{\partial\Phi}{\partial t}=0$$

Modified form; assuming $\frac{\partial\Phi}{\partial t}=0$

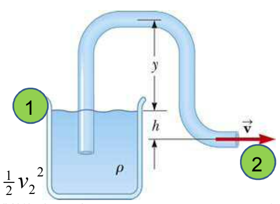
$$\frac{p}{\rho}+U+\tfrac{1}{2}v^2=\text{constant}$$

$p_1=p_2=p_{atm}$

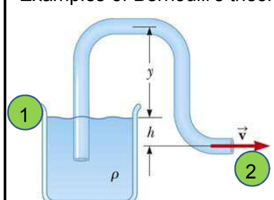
$U_1-U_2=gh$

$v_1\approx 0$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^2$$



Examples of Bernoulli's theorem -- continued



$p_1=p_2=p_{atm}$

$U_1-U_2=gh$

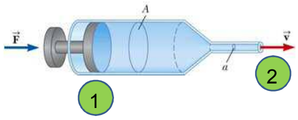
$v_1\approx 0$

$$\frac{p_1}{\rho}+U_1+\tfrac{1}{2}v_1^2=\frac{p_2}{\rho}+U_2+\tfrac{1}{2}v_2^2$$

$$v_2\approx\sqrt{2gh}$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$p_1 = \frac{F}{A} + p_{atm} \qquad p_2 = p_{atm}$$

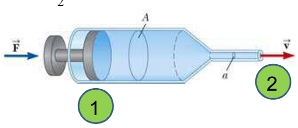
$$U_1 = U_2$$

$$v_1 A = v_2 a \qquad \text{continuity equation}$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^2$$

Examples of Bernoulli's theorem -- continued

$$\frac{p}{\rho} + U + \tfrac{1}{2}v^2 = \text{constant}$$



$$\frac{2F}{A} = v_2^2\left(1 - \left(\frac{a}{A}\right)^2\right)$$

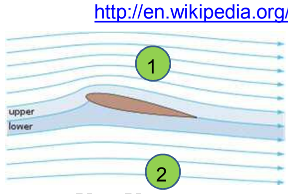
$$v_2 = \sqrt{\frac{2F/A}{1 - \left(\frac{a}{A}\right)^2}}$$

Examples of Bernoulli's theorem – continued

Approximate explanation of airplane lift

Cross section view of airplane wing

http://en.wikipedia.org/wiki/Lift_%28force%29



$$U_1 \approx U_2$$

$$\frac{p_1}{\rho} + U_1 + \tfrac{1}{2}v_1^2 = \frac{p_2}{\rho} + U_2 + \tfrac{1}{2}v_2^2$$

$$p_2 - p_1 = \tfrac{1}{2}\left(v_1^2 - v_2^2\right)$$

Some details on the velocity potential

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{v}) + (\nabla \rho) \cdot \mathbf{v} = 0$$

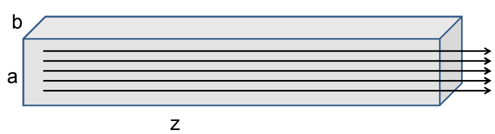
For incompressible fluid : $\rho = (\text{constant})$

$\Rightarrow \nabla \cdot \mathbf{v} = 0$

Irrotational flow : $\nabla \times \mathbf{v} = 0$ $\Rightarrow \mathbf{v} = -\nabla \Phi$

$\Rightarrow \nabla^2 \Phi = 0$

Example – uniform flow



$$\nabla^2 \Phi = 0$$

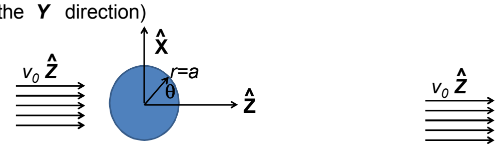
$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Possible solution :

$\Phi = -v_o z$

$\mathbf{v} = -\nabla \Phi = v_o \hat{\mathbf{z}}$

Example – flow around a long cylinder (oriented in the ***Y*** direction)



$$\nabla^2 \Phi = 0$$

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

Laplace equation in cylindrical coordinates

(r, θ, defined in x-z plane; y representing cylinder axis)

$$\nabla^2\Phi = 0 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial y^2}$$

In our case, there is no motion in the y dimension

$$\Rightarrow \Phi(r,\theta,y) = \Phi(r,\theta)$$

From boundary condition : $v_z(r \to \infty) = v_0$

$$\frac{\partial \Phi}{\partial z}(r \to \infty) = -v_0 \qquad \Rightarrow \Phi(r \to \infty, \theta) = -v_0 r \cos\theta$$

Note that : $\frac{\partial^2 \cos\theta}{\partial \theta^2} = -\cos\theta$

Guess form : $\Phi(r,\theta) = f(r)\cos\theta$

Necessary equation for radial function

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial f}{\partial r} - \frac{1}{r^2} f = 0$$

$$f(r) = Ar + \frac{B}{r} \qquad \text{where } A, B \text{ are constants}$$

Boundary condition on cylinder surface :

$$\left.\frac{\partial \Phi}{\partial r}\right|_{r=a} = 0$$

$$\frac{df}{dr}(r = a) = 0 = A - \frac{B}{a^2}$$

$$\Rightarrow B = Aa^2$$

Boundary condition at ∞ : $\Rightarrow A = -v_0$

$$\Phi(r,\theta) = -v_0\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$v_r = -\frac{\partial \Phi}{\partial r} = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

For homework; consider similar boundary value problem for a spherical obstruction

Laplacian in spherical polar coordinates :

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}$$