

**PHY 711 Classical Mechanics and
Mathematical Methods
10-10:50 AM MWF Olin 103**

Plan for Lecture 25:

Rigid body rotational motion (Chap. 5)

- 1. Use of Euler angles as generalized coordinates.**
- 2. Motion of a symmetric top**

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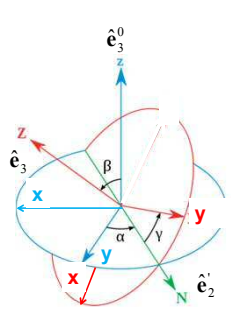
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Date	Topic	Particulars or variations -- Comments	Slide #
7 Wed, 9/11/2013	Chap. 3	Calculus of variations applied to Lagrangians	#6
8 Fri, 9/13/2013	Chap. 3	Lagrangian mechanics	#7
9 Mon, 9/16/2013	Chap. 3 & 6	Lagrangian mechanics	#8
10 Wed, 9/18/2013	Chap. 3 & 6	Lagrangian mechanics	#9
11 Fri, 9/20/2013	Chap. 3 & 6	Lagrangian & Hamiltonian mechanics	#10
12 Mon, 9/23/2013	Chap. 3 & 6	Hamiltonian formalism	#11
13 Wed, 9/25/2013	Chap. 3 & 6	Hamiltonian formalism	#12
14 Fri, 9/27/2013	Chap. 3 & 6	Hamiltonian formalism	#13
15 Mon, 9/30/2013	Chap. 4	Small Oscillations	#14
16 Wed, 10/02/2013	Chap. 4	Small Oscillations	
17 Fri, 10/04/2013	Chap. 4	Small Oscillations	#15
18 Mon, 10/07/2013	Chap. 4 & 7	Small Oscillations and waves	#16
19 Wed, 10/09/2013	Chap. 7	Wave equation	
Fri, 10/11/2013		No class (Fall Break)	
20 Mon, 10/14/2013	Chap. 7	Wave equation (Presentation topic due)	#17
21 Wed, 10/16/2013	Chap. 7	Mathematical methods	#18
22 Fri, 10/18/2013	Chap. 7	Mathematical methods	#19
23 Mon, 10/21/2013	Chap. 5	Rigid rotations	#20
24 Wed, 10/23/2013	Chap. 5	Rigid rotations	#21
25 Fri, 10/25/2013	Chap. 5	Rigid rotations	
Mon, 10/28/2013	No class	Take-home exam	
Wed, 10/30/2013	No class	Take-home exam	
Fri, 11/01/2013	No class	Take-home exam	
26 Mon, 11/04/2013		Take-home exam due	

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$$\tilde{\omega} = \dot{\alpha} \hat{e}_3^0 + \dot{\beta} \hat{e}_2' + \dot{\gamma} \hat{e}_3''$$

$$\tilde{\omega} = [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma] \hat{e}_1 + [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma] \hat{e}_2 + [\dot{\alpha} \cos \beta + \dot{\gamma}] \hat{e}_3$$

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Rotational kinetic energy

$$\begin{aligned}
 T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 \tilde{\omega}_1^2 + \frac{1}{2} I_2 \tilde{\omega}_2^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2 \\
 &= \frac{1}{2} I_1 [\dot{\alpha}(-\sin \beta \cos \gamma) + \dot{\beta} \sin \gamma]^2 \\
 &\quad + \frac{1}{2} I_2 [\dot{\alpha}(\sin \beta \sin \gamma) + \dot{\beta} \cos \gamma]^2 \\
 &\quad + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2
 \end{aligned}$$

If $I_1 = I_2$:

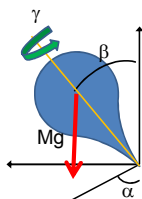
$$T(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) = \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2$$

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Motion of a symmetric top under the influence of the torque of gravity:



$$\begin{aligned}
 L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \\
 &\quad \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta
 \end{aligned}$$

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$$\begin{aligned}
 L(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) &= \frac{1}{2} I_1 (\dot{\alpha}^2 \sin^2 \beta + \dot{\beta}^2) + \\
 &\quad \frac{1}{2} I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]^2 - Mgl \cos \beta
 \end{aligned}$$

Constants of the motion :

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + V_{\text{eff}}(\beta)$$

$$L(\beta, \dot{\beta}) = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\gamma^2}{2I_3} - Mgl \cos \beta$$

$$V_{\text{eff}}(\beta) = \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

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$$E = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2}{2I_3} + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

Stable/unstable solutions near $\beta=0$

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Suppose $p_\alpha = p_\gamma$ and $\beta \approx 0$

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

$$E' \approx \frac{1}{2} I_1 \dot{\beta}^2 + \frac{p_\gamma^2 (1 - 1 + \frac{1}{2} \beta^2)^2}{\beta^2} + Mgl (1 - \frac{1}{2} \beta^2)$$

$$\approx \frac{1}{2} I_1 \dot{\beta}^2 + \left(\frac{p_\gamma^2}{8I_1} - \frac{Mgl}{2} \right) \beta^2 + Mgl$$

\Rightarrow Stable solution if $p_\gamma \geq \sqrt{4MglI_1}$

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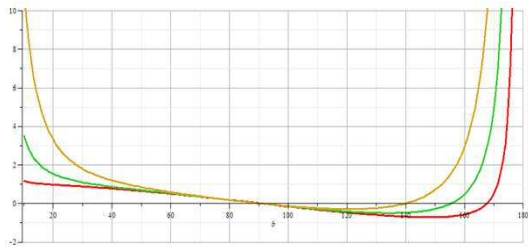
More general case:

$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$

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More general case:

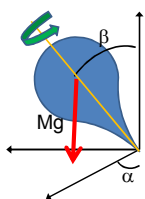
$$E' = E - \frac{p_\gamma^2}{2I_3} = \frac{1}{2} I_1 \dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + Mgl \cos \beta$$



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Constants of the motion :

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \dot{\alpha} \sin^2 \beta + I_3 [\dot{\alpha} \cos \beta + \dot{\gamma}] \cos \beta$$

$$= I_1 \dot{\alpha} \sin^2 \beta + p_\gamma \cos \beta$$

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