PHY 711 Classical Mechanics and Mathematical Methods
10-10:50 AM MWF Olin 103

Plan for Lecture 23:

Rotational motion (Chapter 5)

1. Rigid body motion

2. Moment of inertia tensor

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6	Mon, 9/09/2013	Chap. 3	Calculus of variations -- continued	
7	Wed, 9/11/2013	Chap. 3	Calculus of variations applied to Lagrangians	#6
8	Fri, 9/13/2013	Chap. 3	Lagrangian mechanics	#7
9	Mon, 9/16/2013	Chap. 3 & 6	Lagrangian mechanics	#8
10	Wed, 9/18/2013	Chap. 3 & 6	Lagrangian mechanics	#9
11	Fri, 9/20/2013	Chap. 3 & 6	Lagrangian & Hamiltonian mechanics	#10
12	Mon, 9/23/2013	Chap. 3 & 6	Hamiltonian formalism	#11
13	Wed, 9/25/2013	Chap. 3 & 6	Hamiltonian formalism	#12
14	Fri, 9/27/2013	Chap. 3 & 6	Hamiltonian formalism	#13
15	Mon, 9/30/2013	Chap. 4	Small Oscillations	#14
16	Wed, 10/02/2013	Chap. 4	Small Oscillations	
17	Fri, 10/04/2013	Chap. 4	Small Oscillations	#15
18	Mon, 10/07/2013	Chap. 4 & 7	Small Oscillations and waves	#16
19	Wed, 10/09/2013	Chap. 7	Wave equation	
	Fri, 10/11/2013		No class (Fall Break)	
20	Mon, 10/14/2013	Chap. 7	Wave equation (Presentation topic due)	#17
21	Wed, 10/16/2013	Chap. 7	Mathematical methods	#18
22	Fri, 10/18/2013	Chap. 7	Mathematical methods	#19
23	Mon, 10/21/2013	Chap. 5	Rigid rotations	#20
24	Wed, 10/23/2013	Chap. 5	Rigid rotations	
25	Fri, 10/25/2013	Chap. 5	Rigid rotations	
	Mon, 10/28/2013	No class	Take-home exam	
	Wed, 10/30/2013	No class	Take-home exam	
	Fri, 11/01/2013	No class	Take-home exam	
26	Mon, 11/04/2013		Take-home exam due	

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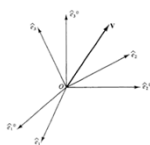The physics of rigid body motion; body fixed frame vs inertial frame:

Figure 6.1 Transformation to a rotating co-ordinate system.

Let **V** be a general vector, e.g., the position of a particle. This vector can be characterized by its components with respect to either orthonormal triad. Thus we can write

$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 \qquad (6.1a)$$

$$\mathbf{V} = \sum_{i=1}^{3} V_i \hat{e}_i \qquad (6.1b)$$

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Comparison of analysis in "inertial frame" versus "non-inertial frame"

Denote by $\hat{e}_i^0$ a fixed coordinate system

Denote by $\hat{e}_i$ a moving coordinate system

$$\mathbf{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 = \sum_{i=1}^{3} V_i \hat{e}_i$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \sum_{i=1}^{3} \frac{dV_i^0}{dt}\hat{e}_i^0 = \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\text{Define}: \left(\frac{d\mathbf{V}}{dt}\right)_{body} \equiv \sum_{i=1}^{3} \frac{dV_i}{dt}\hat{e}_i$$
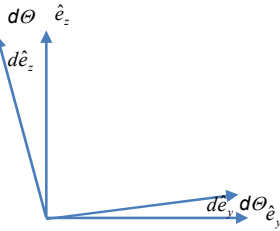
$$\Rightarrow \left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

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Properties of the frame motion (rotation):

$$d\hat{e}_y = d\Theta\hat{e}_z$$
$$d\hat{e}_z = -d\Theta\hat{e}_y$$
$$\Rightarrow d\hat{\mathbf{e}} = d\Theta \times \hat{\mathbf{e}}$$
$$\frac{d\hat{\mathbf{e}}}{dt} = \frac{d\Theta}{dt} \times \hat{\mathbf{e}}$$
$$\frac{d\hat{\mathbf{e}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}$$

$$\begin{pmatrix} d\hat{e}_y \\ d\hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos(d\Theta) & \sin(d\Theta) \\ -\sin(d\Theta) & \cos(d\Theta) \end{pmatrix}\begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix} - \begin{pmatrix} \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$

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$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}$$

Effects on acceleration:

$$\left(\frac{d}{dt}\frac{d\mathbf{V}}{dt}\right)_{inertial} = \left(\left(\frac{d}{dt}\right)_{body} + \boldsymbol{\omega} \times\right)\left\{\left(\frac{d\mathbf{V}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{V}\right\}$$
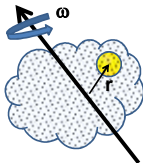
$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{inertial} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{body} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{body} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{V} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{V}$$

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Kinetic energy of rigid body :

$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{r}$$

$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega}\times\mathbf{r}$$

$$T = \sum_p \frac{1}{2}m_p v_p^2 = \sum_p \frac{1}{2}m_p\left(\left|\boldsymbol{\omega}\times\mathbf{r}_p\right|\right)^2$$

$$= \sum_p \frac{1}{2}m_p\left(\boldsymbol{\omega}\times\mathbf{r}_p\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_p\right)$$

$$= \sum_p \frac{1}{2}m_p\left[\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right)\left(\mathbf{r}_p\cdot\mathbf{r}_p\right) - \left(\mathbf{r}_p\cdot\boldsymbol{\omega}\right)^2\right]$$

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$$T = \sum_p \frac{1}{2}m_p\left[\left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right)\left(\mathbf{r}_p\cdot\mathbf{r}_p\right) - \left(\mathbf{r}_p\cdot\boldsymbol{\omega}\right)^2\right]$$

$$= \boldsymbol{\omega}\cdot\vec{\mathbf{I}}\cdot\boldsymbol{\omega}$$

Moment of inertia tensor :

$$\vec{\mathbf{I}} \equiv \sum_p m_p\left(\mathbf{1}r_p^2 - \mathbf{r}_p\mathbf{r}_p\right) \qquad \text{(dyad notation)}$$
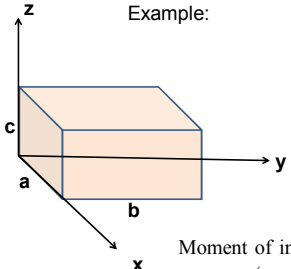
Matrix notation :

$$\vec{\mathbf{I}} \equiv \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{ij} \equiv \sum_p m_p\left(\delta_{ij}r_p^2 - r_{pi}r_{pj}\right)$$

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Example:

Moment of inertia tensor :

$$\vec{\mathbf{I}} = M\begin{pmatrix} \frac{1}{3}\left(b^2+c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2+c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2+b^2\right) \end{pmatrix}$$

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Properties of moment of inertia tensor:
> Symmetric matrix ➜ real eigenvalues I_1, I_2, I_3
> ➜ orthogonal eigenvectors

$$\vec{\mathbf{I}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad i = 1,2,3$$

Moment of inertia tensor :

$$\vec{\mathbf{I}} = M \begin{pmatrix} \frac{1}{3}\left(b^2 + c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2 + c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2 + b^2\right) \end{pmatrix}$$

For $\quad a = b = c$:

$$I_1 = \frac{1}{6}Ma^2 \quad I_2 = \frac{11}{12}Ma^2 \quad I_3 = \frac{11}{12}Ma^2$$

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Changing origin of rotation

$$I_{ij} \equiv \sum_p m_p \left(\delta_{ij} r_p^2 - r_{pi} r_{pj}\right)$$

$$I'_{ij} \equiv \sum_p m_p \left(\delta_{ij} r'^2_p - r'_{pi} r'_{pj}\right)$$

$$\mathbf{r}'_p = \mathbf{r}_p + \mathbf{R}$$

Define the center of mass :

$$\mathbf{r}_{CM} = \frac{\sum_p m_p \mathbf{r}_p}{\sum_p m_p} \equiv \frac{\sum_p m_p \mathbf{r}_p}{M}$$

$$I'_{ij} = I_{ij} + M\left(R^2 \delta_{ij} - R_i R_j\right) + M\left(2\mathbf{r}_{CM} \cdot \mathbf{R} \delta_{ij} - r_{CMi} R_j - R_i r_{CMj}\right)$$

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$$I'_{ij} = I_{ij} + M\left(R^2 \delta_{ij} - R_i R_j\right) + M\left(2\mathbf{r}_{CM} \cdot \mathbf{R} \delta_{ij} - r_{CMi} R_j - R_i r_{CMj}\right)$$

Suppose that $\mathbf{R} = -\frac{a}{2}\hat{\mathbf{x}} - \frac{b}{2}\hat{\mathbf{y}} - \frac{c}{2}\hat{\mathbf{z}}$
and $\quad \mathbf{r}_{CM} = -\mathbf{R}$

$$I'_{ij} = I_{ij} - M\left(R^2 \delta_{ij} - R_i R_j\right)$$

$$\vec{\mathbf{I}}' = M \begin{pmatrix} \frac{1}{3}\left(b^2 + c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{3}\left(a^2 + c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{3}\left(a^2 + b^2\right) \end{pmatrix}$$

$$- M \begin{pmatrix} \frac{1}{4}\left(b^2 + c^2\right) & -\frac{1}{4}ab & -\frac{1}{4}ac \\ -\frac{1}{4}ab & \frac{1}{4}\left(a^2 + c^2\right) & -\frac{1}{4}bc \\ -\frac{1}{4}ac & -\frac{1}{4}bc & \frac{1}{4}\left(a^2 + b^2\right) \end{pmatrix}$$

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$$\ddot{\mathbf{I}}' = M \begin{pmatrix} \frac{1}{12}\left(b^2 + c^2\right) & 0 & 0 \\ 0 & \frac{1}{12}\left(a^2 + c^2\right) & 0 \\ 0 & 0 & \frac{1}{12}\left(a^2 + b^2\right) \end{pmatrix}$$

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Descriptions of rotation about a given origin

For general coordinate system

$$T = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$$

For (body fixed) coordinate system that diagonalizes
moment of inertia tensor :

$$\ddot{\mathbf{I}}\cdot\hat{\mathbf{e}}_i = I_i\hat{\mathbf{e}}_i \qquad i = 1,2,3$$

$$\boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\Rightarrow T = \frac{1}{2}\sum_i I_i\tilde{\omega}_i^2$$

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Descriptions of rotation about a given origin -- continued
Time rate of change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L}$$

For (body fixed) coordinate system that diagonalizes
moment of inertia tensor :

$$\ddot{\mathbf{I}}\cdot\hat{\mathbf{e}}_i = I_i\hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1\hat{\mathbf{e}}_1 + \tilde{\omega}_2\hat{\mathbf{e}}_2 + \tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1\tilde{\omega}_1\hat{\mathbf{e}}_1 + I_2\tilde{\omega}_2\hat{\mathbf{e}}_2 + I_3\tilde{\omega}_3\hat{\mathbf{e}}_3$$

$$\frac{d\mathbf{L}}{dt} = I_1\dot{\tilde{\omega}}_1\hat{\mathbf{e}}_1 + I_2\dot{\tilde{\omega}}_2\hat{\mathbf{e}}_2 + I_3\dot{\tilde{\omega}}_3\hat{\mathbf{e}}_3 + \tilde{\omega}_2\tilde{\omega}_3\left(I_3 - I_2\right)\hat{\mathbf{e}}_1$$

$$+ \tilde{\omega}_3\tilde{\omega}_1\left(I_1 - I_3\right)\hat{\mathbf{e}}_2 + \tilde{\omega}_1\tilde{\omega}_2\left(I_2 - I_1\right)\hat{\mathbf{e}}_3$$

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Descriptions of rotation about a given origin -- continued

Time rate of change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

For (body fixed) coordinate system that diagonalizes
moment of inertia tensor :

$$\bar{\bar{\mathbf{I}}} \cdot \hat{\mathbf{e}}_i = I_i \hat{\mathbf{e}}_i \qquad \boldsymbol{\omega} = \tilde{\omega}_1 \hat{\mathbf{e}}_1 + \tilde{\omega}_2 \hat{\mathbf{e}}_2 + \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\mathbf{L} = I_1 \tilde{\omega}_1 \hat{\mathbf{e}}_1 + I_2 \tilde{\omega}_2 \hat{\mathbf{e}}_2 + I_3 \tilde{\omega}_3 \hat{\mathbf{e}}_3$$

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1$$
$$+ \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3$$

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Descriptions of rotation about a given origin -- continued

Note that the torque equation

$$\frac{d\mathbf{L}}{dt} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\tau}$$

is very difficult to solve directly in the body fixed frame.
For $\boldsymbol{\tau} = 0$ we can solve the Euler equations :

$$\frac{d\mathbf{L}}{dt} = I_1 \dot{\tilde{\omega}}_1 \hat{\mathbf{e}}_1 + I_2 \dot{\tilde{\omega}}_2 \hat{\mathbf{e}}_2 + I_3 \dot{\tilde{\omega}}_3 \hat{\mathbf{e}}_3 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) \hat{\mathbf{e}}_1$$
$$+ \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) \hat{\mathbf{e}}_2 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) \hat{\mathbf{e}}_3 = 0$$

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Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$
$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$
$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for symmetric top -- $I_2 = I_1$:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_1) = 0$$
$$I_1 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$
$$I_3 \dot{\tilde{\omega}}_3 = 0 \qquad \Rightarrow \tilde{\omega}_3 = \text{(constant)}$$

Define : $\quad \Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1} \qquad \dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega$
$$\dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

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Solution of Euler equations for a symmetric top -- continued

$$\dot{\tilde{\omega}}_1 = -\tilde{\omega}_2 \Omega \qquad \dot{\tilde{\omega}}_2 = \tilde{\omega}_1 \Omega$$

where $\Omega \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_1}$

Solution : $\quad \tilde{\omega}_1(t) = A\cos(\Omega t + \varphi)$

$$\tilde{\omega}_2(t) = A\sin(\Omega t + \varphi)$$

$$T = \frac{1}{2} \sum_i I_i \tilde{\omega}_i^2 = \frac{1}{2} I_1 A^2 + \frac{1}{2} I_3 \tilde{\omega}_3^2$$

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Euler equations for rotation in body fixed frame :

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$
$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$
$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Solution for asymmetric top $- - I_3 \neq I_2 \neq I_1$:

$$I_1 \dot{\tilde{\omega}}_1 + \tilde{\omega}_2 \tilde{\omega}_3 (I_3 - I_2) = 0$$
$$I_2 \dot{\tilde{\omega}}_2 + \tilde{\omega}_3 \tilde{\omega}_1 (I_1 - I_3) = 0$$
$$I_3 \dot{\tilde{\omega}}_3 + \tilde{\omega}_1 \tilde{\omega}_2 (I_2 - I_1) = 0$$

Suppose : $\quad \dot{\tilde{\omega}}_3 \approx 0 \qquad$ Define : $\Omega_1 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_2}{I_1}$

Define : $\Omega_2 \equiv \tilde{\omega}_3 \dfrac{I_3 - I_1}{I_2}$

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